The Gysin exact sequence for S^1-equivariant symplectic homology

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Abstract

We define S^1-equivariant symplectic homology for symplectically aspherical manifolds with contact boundary, using a Floer-type construction first proposed by Viterbo. We show that it is related to the usual symplectic homology by a Gysin exact sequence. As an important ingredient of the proof, we define a parametrized version of symplectic homology, corresponding to families of Hamiltonian functions indexed by a finite dimensional smooth parameter space. We define a parametrized version of the Robbin-Salamon index, which gives the grading for these new versions of symplectic homology. We indicate several applications and ramifications of our constructions.

Contents

1 Introduction

TOPOLOGICAL BACKGROUND. Given an oriented fibration $S^1 \hookrightarrow M \xrightarrow{\pi} B$, the homology groups of the base and total space are related by the **Gysin exact sequence**

$$\ldots \to H_k(M) \xrightarrow{\pi_*} H_k(B) \xrightarrow{D} H_{k-2}(B) \to H_{k-1}(M) \to \ldots \tag{1.1}$$

Here D is the cap-product with the Euler class of the fibration and is equal to the differential d^2 of the Leray-Serre spectral sequence [16, Example 5.C].

A particular case of the above construction is the following. Assume M carries an S^1-action and define the S^1-equivariant homology $H_*^{S^1}(M)$ by

$$H_*^{S^1}(M) := H_*(M_{S^1}), \qquad M_{S^1} := M \times_{S^1} ES^1,$$

where ES^1 is a contractible space on which S^1 acts freely. Since S^1 acts freely on $M \times ES^1$, the projection $M \times ES^1 \to M_{S^1}$ is an S^1-fibration and the exact sequence (1.1) becomes

$$\ldots \to H_k(M) \to H_k^{S^1}(M) \xrightarrow{D} H_{k-2}^{S^1}(M) \to H_{k-1}(M) \to \ldots \tag{1.2}$$

We call this the **Gysin exact sequence for S^1-equivariant homology**. Two relevant instances of this construction are the following:

(i) If the action of S^1 on M is free then $H_*^{S^1}(M) \simeq H_*(M/S^1)$ and the Gysin exact sequence for S^1-equivariant homology is the Gysin exact sequence for the fibration $S^1 \hookrightarrow M \to M/S^1$.

(ii) We denote $BS^1 := ES^1/S^1$. Taking the model of ES^1 to be $S^\infty := \lim_{N\to\infty} S^{2N+1}$, with S^{2N+1} the unit sphere in $\mathbb{C}^{N+1}$, we see that $BS^1 \simeq \mathbb{C}P^\infty$.

Now, if S^1 acts trivially on M, then $H_*^{S^1}(M) \simeq H_*(M) \otimes H_*(BS^1)$ and (1.2) becomes

$$\ldots \xrightarrow{0} H_k(M) \xrightarrow{i} \bigoplus_{m\geq 0} H_{k-2m}(M) \xrightarrow{p} \bigoplus_{m\geq 1} H_{k-2m}(M) \xrightarrow{0} H_{k-1}(M) \to \ldots$$

Here i and p are the obvious inclusion and projection.

Main results. This paper is concerned with a Floer homology long exact sequence of Gysin type. Let (W, ω) be a symplectic manifold with contact type boundary satisfying

$$\int_{T^2} f^*\omega = 0 \quad \text{for all smooth } f : T^2 \to W. \tag{1.3}$$

Our main class of examples consists of exact symplectic manifolds. Let a be a free homotopy class of loops in W. One can define in this situation **symplectic homology groups** $SH_*^a(W)$ and $\boldsymbol{S^1}$**-equivariant symplectic homology groups** $SH_*^{a,S^1}(W)$, as well as variants $SH_*^+(W)$, $SH_*^{+,S^1}(W)$ truncated in positive values of the action functional when $a = 0$. The original definition is due to Viterbo [27] and we refer to §4.2 for the details of the construction. Our first result is the following.

Theorem 1.1. *The symplectic homology groups fit into an exact sequence of Gysin type (we allow $a = +$)*

$$\ldots \to SH_k^a(W) \to SH_k^{a,S^1}(W) \xrightarrow{D} SH_{k-2}^{a,S^1}(W) \to SH_{k-1}^a(W) \to \ldots \tag{1.4}$$

As a matter of fact, we prove in [6] that the above Gysin exact sequence for $a = +$ is isomorphic to the long exact sequence of [4], relating $SH_*^+(W)$ with the linearized contact homology of the filled contact manifold ∂W.

In the case $a = 0$, the symplectic homology groups

$$SH_*(W) := SH_*^0(W), \qquad SH_*^{S^1}(W) := SH_*^{0,S^1}(W)$$

also fit into tautological long exact sequences [27]

$$\ldots \to SH_{*+1}^+(W) \to H_{*+n}(W, \partial W) \to SH_*(W) \to SH_*^+(W) \to \ldots\ , \tag{1.5}$$

$$\ldots \to SH_{*+1}^{+,S^1}(W) \to H_{*+n}^{S^1}(W, \partial W) \to SH_*^{S^1}(W) \to SH_*^{+,S^1}(W) \to \ldots \tag{1.6}$$

Here the S^1-equivariant homology of the pair $(W, \partial W)$ is considered with respect to the trivial action of S^1. Our next result is that the Gysin exact sequence is compatible with these tautological exact sequences.

Theorem 1.2. *There is a commutative diagram whose rows and columns are, respectively, the tautological and Gysin exact sequences*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & SH^+_{k+1} \arrow[r] \arrow[d] & H_{k+n} \arrow[r] \arrow[d] & SH_k \arrow[r] \arrow[d] & SH^+_k \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & SH^{+,S^1}_{k+1} \arrow[r] \arrow[d] & H^{S^1}_{k+n} \arrow[r] \arrow[d] & SH^{S^1}_k \arrow[r] \arrow[d] & SH^{+,S^1}_k \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & SH^{+,S^1}_{k-1} \arrow[r] \arrow[d] & H^{S^1}_{k+n-2} \arrow[r] \arrow[d] & SH^{S^1}_{k-2} \arrow[r] \arrow[d] & SH^{+,S^1}_{k-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & SH^+_k \arrow[r] \arrow[d] & H_{k+n-1} \arrow[r] \arrow[d] & SH_{k-1} \arrow[r] \arrow[d] & SH^+_{k-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{1.7}$$

Examples. We discuss the consequences of our main theorems for two important classes of symplectic manifolds.

Cotangent bundles. Let L be a closed oriented Riemannian manifold, and denote by ΛL the free loop space of L. We consider the symplectic manifold $W = DT^*L = \{p \in T^*L \, : \, \|p\| \leq 1\}$. It was proved by Viterbo [28] that

$$SH_*(DT^*L) \simeq H_*(\Lambda L), \qquad SH^{S^1}_*(DT^*L) \simeq H^{S^1}_*(\Lambda L).$$

Alternative proofs for the first isomorphism are due to Abbondandolo and Schwarz [1], respectively to Salamon and Weber [23]. Our proof of Theorem 1.1 can be combined with the methods of [1] in order to prove that the long exact sequence (1.4) is isomorphic to the Gysin sequence for ΛL, namely

$$\cdots \longrightarrow H_*(\Lambda L) \xrightarrow{E} H^{S^1}_*(\Lambda L) \xrightarrow{D} H^{S^1}_{*-2}(\Lambda L) \xrightarrow{M} H_{*-1}(\Lambda L) \longrightarrow \cdots \tag{1.8}$$

Similarly, for $a = +$, we obtain the Gysin sequence of the pair $(\Lambda^0 L, L)$, where $\Lambda^0 L$ is the component of free contractible loops in L.

Subcritical Stein manifolds. A subcritical Stein manifold is a complex manifold (W, J), of complex dimension n, endowed with a pluri-subharmonic function $\phi : W \to \mathbb{R}$, satisfying the following conditions: (i) the boundary ∂W is a regular level set of ϕ along which $\vec{\nabla}\phi$ points outwards; (ii) ϕ is Morse

and the index of all its critical points is strictly smaller than n. The complex structure J is compatible with the natural symplectic form $\omega_\phi := -d(d\phi \circ J)$.

It was proved by Cieliebak [8] that $SH_*(W) = 0$. His proof can be adapted in a straightforward way in order to show that $SH_*^{S^1}(W) = 0$. However, this fact follows also from Theorem 1.1 in the case $c_1(W) = 0$.

Corollary 1.3. *Assume W is a subcritical Stein manifold with $c_1(W) = 0$. Then we have $SH_*^{S^1}(W) = 0$ and there is an isomorphism of exact sequences*

$$\begin{tikzcd}
\cdots \arrow[r] & SH_*^+(W) \arrow[r] \arrow[d, "\simeq"'] & SH_*^{+,S^1}(W) \arrow[r, "D"] \arrow[d, "\simeq"'] & SH_{*-2}^{+,S^1}(W) \arrow[r] \arrow[d, "\simeq"'] & SH_{*-1}^+(W) \arrow[r] \arrow[d, "\simeq"'] & \cdots \\
\cdots \arrow[r, "0"] & H_{*+n-1}(W,\partial W) \arrow[r] & H_{*+n-1}^{S^1}(W,\partial W) \arrow[r] & H_{*+n-3}^{S^1}(W,\partial W) \arrow[r, "0"] & H_{*+n-2}(W,\partial W) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Applying Theorem 1.1 we obtain that $SH_k^{S^1}(W) \simeq SH_{k-2}^{S^1}(W)$ for all $k \in \mathbb{Z}$. It was proved by M.-L. Yau [29] that one can choose the plurisubharmonic function ϕ so that the Conley-Zehnder indices of all closed characteristics on ∂W are positive. It follows from the definition of S^1-equivariant symplectic homology that the underlying chain complex is zero if the degree is small enough (one can use "split" Hamiltonians as in the proof of Lemma 4.7). Reasoning by induction, it follows that $SH_*^{S^1}(W) = 0$. The isomorphism of exact sequences follows immediately from Theorem 1.2, since the columns involving SH_* and $SH_*^{S^1}$ vanish identically. $\square$

Algebraic Weinstein conjecture. Following Viterbo [27], we say that W satisfies the *Strong Algebraic Weinstein Conjecture (SAWC)* if the map

$$H_{2n}(W,\partial W) \to SH_n(W)$$

vanishes. Let $\mu_{2n} \in H_{2n}(W,\partial W)$ be the fundamental class and u_k be a generator of $H_{2k}(BS^1)$, $k \geq 0$. We say that W satisfies the *Strong Equivariant Algebraic Weinstein Conjecture (EWC)* if, for all $k \geq 0$, the element $\mu_{2n} \otimes u_k$ lies in the kernel of the map

$$H_{2n+2k}^{S^1}(W,\partial W) \to SH_{n+2k}^{S^1}(W).$$

Our next result clarifies the relationship between $SAWC$ and EWC, which are the two key notions in Viterbo's fundamental paper [27].

Corollary 1.4. *$SAWC \Longrightarrow EWC$.*

Proof. We first note that $SAWC$ is equivalent to the vanishing of $SH_*(W)$. This follows from the fact that $SH_*(W)$ is a ring with unit [17], and the unit is the image of the fundamental class μ_{2n} under the map $H_{2n}(W, \partial W) \to SH_n(W)$ [26].

We now consider the top middle square in the commutative diagram (1.7) of Theorem 1.2. Since $\mu_{2n} \otimes u_0$ is the image of μ_{2n} under the injection $H_{2n} \to H_{2n}^{S^1}$, it follows that $\mu_{2n} \otimes u_0$ is in the kernel of $H_{2n}^{S^1} \to SH_n^{S^1}$. We now prove by induction that $\mu_{2n} \otimes u_k$ is in the kernel of $H_{2n+2k}^{S^1} \to SH_{n+2k}^{S^1}$. This follows from the middle square in the commutative diagram (1.7), using that $\mu_{2n} \otimes u_{k+1}$ is sent to $\mu_{2n} \otimes u_k$ by the map $H_{2n+2k+2}^{S^1} \to H_{2n+2k}^{S^1}$, and the fact that $SH_{n+2k+2}^{S^1} \to SH_{n+2k}^{S^1}$ is an isomorphism. $\square$

Remark 1.5. The same argument as above shows that, under the assumption $SAWC$, the maps $H_{k+n}^{S^1} \to SH_k^{S^1}$ vanish for all $k \in \mathbb{Z}$.

The parametrized Robbin-Salamon index. Given a smooth manifold X with an S^1-action, the S^1-equivariant homology $H_*^{S^1}(X)$ can be realized as S^1-invariant Morse homology on $X \times ES^1$, which in turn can be approximated by $X \times S^{2N+1}$, $N \to \infty$. In analogy, S^1-equivariant symplectic homology is defined as an S^1-invariant Floer theory for a parametrized action functional $\mathcal{A}_H : C^\infty(S^1, W) \times S^{2N+1} \to \mathbb{R}$ which, on contractible loops, is of the form

$$\mathcal{A}_H(\gamma, \lambda) := -\int_{D^2} \sigma^*\omega - \int_{S^1} H(\theta, \gamma(\theta), \lambda)\, d\theta,$$

where $\sigma : D^2 \to W$ is a capping disc for γ. Here $H : S^1 \times W \times S^{2N+1} \to \mathbb{R}$ is an S^1-invariant family of Hamiltonians, meaning that $H(\theta + \tau, x, \tau\lambda) = H(\theta, x, \lambda)$ for all $\tau \in S^1$. This condition ensures that $\mathcal{A}_H$ is invariant with respect to the diagonal action of S^1 on $C^\infty(S^1, W) \times S^{2N+1}$, given by $\tau \cdot (\gamma, \lambda) := (\gamma(\cdot - \tau), \tau\lambda)$.

The corresponding chain complex is generated by S^1-orbits of critical points of $\mathcal{A}_H$, which are pairs (γ, λ) such that γ is a 1-periodic orbit of the Hamiltonian vector field of $H(\cdot, \cdot, \lambda)$, and $\int_{S^1} \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)\, d\theta = 0$ (see §3.1). In order to associate a grading to these generators, we define in §3.2 a parametrized version of the Robbin-Salamon index.

The construction is valid for an arbitrary finite-dimensional smooth parameter space Λ. Given a parametrized Hamiltonian $H : S^1 \times W \times \Lambda \to \mathbb{R}$, we extend it to $\widetilde{H} : S^1 \times W \times T^*\Lambda \to \mathbb{R}$ by the formula

$$\widetilde{H}(t, x, (\lambda, p)) := H(t, x, \lambda).$$

We then have $X_{\widetilde{H}} = X_{H_\lambda} + \frac{\partial H}{\partial \lambda}\frac{\partial}{\partial p}$, with $H_\lambda := H(\cdot,\cdot,\lambda)$. A 1-periodic orbit of $X_{\widetilde{H}}$ has the form $(\gamma(\cdot), p(\cdot), \lambda)$, with γ a 1-periodic orbit of X_{H_λ} and $p(t) = p(0) + \int_0^t \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)\, d\theta$. The closing condition $p(1) = p(0)$ is equivalent to the condition $\int_0^1 \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)\, d\theta = 0$, while $p(0) \in T^*_\lambda \Lambda$ can be chosen arbitrarily. Thus (nondegenerate) critical points of the parametrized action functional for H are in one-to-one bijective correspondence with (Morse-Bott) families of 1-periodic orbits of $\widetilde{H}$, of dimension $\dim \Lambda = \dim T^*_\lambda\Lambda$.

Definition 1.6. *The* parametrized Robbin-Salamon index $\mu(\gamma, \lambda)$ *of a critical point* $(\gamma(\cdot), \lambda)$ *for the parametrized Hamiltonian* H *is the Robbin-Salamon index [20] of one of the corresponding* 1*-periodic orbits* $(\gamma(\cdot), p(\cdot), \lambda)$ *of* $\widetilde{H}$*.*

The construction given in §3.2 is phrased directly in terms of the Hamiltonian H, rather than in terms of $\widetilde{H}$ (see (3.15)). The properties of the parametrized Robbin-Salamon index are proved in Appendix A, and they amount to the study of paths of symplectic matrices of a special form. The dimension of the moduli spaces of trajectories connecting a pair of critical points is given by the difference of the indices of these critical points (see Theorem 3.5).

Ramifications. We now present several directions of investigation which are related to the present paper.

Algebraic structures. The Gysin exact sequence (1.4) can be used to define algebraic operations in (S^1-equivariant) symplectic homology.

As already mentioned in the proof of Corollary 1.4, symplectic homology $SH_*(W)$ is a unitary ring, with the pair-of-pants product. This is described by Seidel [26], and was used in a crucial way by McLean [17] in his construction of exotic affine $\mathbb{R}^{2n}$'s. We denote the pair of pants product by

$$\bullet : SH_k(W) \otimes SH_\ell(W) \longrightarrow SH_{k+\ell-n}(W).$$

Let us write the Gysin exact sequence (1.4) as

$$\cdots \longrightarrow SH_*(W) \xrightarrow{E} SH_*^{S^1}(W) \xrightarrow{D} SH_{*-2}^{S^1}(W) \xrightarrow{M} SH_{*-1}(W) \longrightarrow \cdots$$

The notation is motivated by the isomorphism with the exact sequence (1.8) in the case $W = DT^*L$. The letters M and E stand for "mark" and "erase", in the terminology of Chas and Sullivan [7]. It was proved by Abbondandolo

and Schwarz [2] that, in the case $W = DT^*L$, the pair-of-pants product is identified with the Chas-Sullivan loop product [7].

Inspired by Chas and Sullivan [7], we formulate the following definitions and claims, which we will prove in a forthcoming paper.

— The map

$$\Delta : SH_*(W) \to SH_{*+1}(W), \qquad \Delta := M \circ E$$

is a *Batalin-Vilkovisky (BV) operator*, in the sense that $\Delta^2 = 0$, and

$$\{\cdot, \cdot\} : SH_k(W) \otimes SH_\ell(W) \to SH_{k+\ell-n+1}(W),$$

$$\{a, b\} \ := \ \pm \Delta(a \bullet b) \pm a \bullet \Delta(b) \pm b \bullet \Delta(a)$$

is a bracket on $SH_*(W)$ (called *the loop bracket*).

— The map

$$[\cdot, \cdot] : SH_k^{S^1}(W) \otimes SH_\ell^{S^1}(W) \to SH_{k+\ell-n+2}^{S^1}(W)$$

$$[a, b] := \pm E(M(a) \bullet M(b))$$

is a bracket on $SH_*^{S^1}(W)$ (called *the string bracket*).

We give a chain-level description of Δ in Remark 5.7. The above claims are analogous to Theorems 4.7, 5.4, and 6.1 of [7]. The string bracket can be further generalized as follows. Any operation

$$\widetilde{\sigma} : SH_*^{\otimes k} \to SH_*, k \geq 2$$

yields an operation

$$\sigma := E \circ \widetilde{\sigma} \circ M^{\otimes k} : (SH_*^{S^1})^{\otimes k} \to SH_*^{S^1}.$$

One particular case is $\widetilde{\sigma} := \bullet^{\otimes k-1}$, $k \geq 2$, which yields higher-order operations on $SH_*^{S^1}$ analogous to the ones of [7, Theorem 6.2].

The range of applications of such operations depends on their explicit knowledge in particular situations (e.g. cotangent bundles). However, the Chas-Sullivan string operations are only beginning to be understood by topologists (see the work of Felix, Thomas, and Vigué-Poirrier [9, 10]).

It should also be possible to describe these operations directly in terms of holomorphic curves. Such a construction is sketched by Seidel in [26].

Relation to Hochschild and cyclic homology. Paul Seidel has conjectured in [25] that, given an exact Lefschetz fibration $E \to D$ over the disc, the symplectic homology of E is isomorphic to the Hochschild homology of a certain A_∞-category $\mathcal{C}$ built from the vanishing cycles of E:

$$SH_*(E) \simeq HH_*(\mathcal{C}).$$

It is implicit in [25] that there is an equivariant version of this conjectural isomorphism, namely that the S^1-equivariant symplectic homology of E is isomorphic to the cyclic homology of $\mathcal{C}$:

$$SH_*^{S^1}(E) \simeq HC_*(\mathcal{C}).$$

On the other hand, Hochschild and cyclic homology are related by the Connes exact sequence

$$\ldots \to HH_k(\mathcal{C}) \to HC_k(\mathcal{C}) \xrightarrow{D} HC_{k-2}(\mathcal{C}) \to HH_{k-1}(\mathcal{C}) \to \ldots \tag{1.9}$$

We conjecture that the two previous isomorphisms are such that the Gysin exact sequence (1.4) and the Connes exact sequence (1.9) are isomorphic. This fits with the general philosophy that the Gysin exact sequence for S^1-equivariant homology of certain topological spaces is isomorphic to the Connes exact sequence of suitable algebras (a good reference is Loday's book [15], in particular [15, Theorem 7.2.3]).

Relation to Givental's point of view. Given a closed symplectic manifold X, Givental defined in [13] a D-module structure on $H^*(X;\mathbb{C}) \otimes \Lambda_{Nov} \otimes \mathbb{C}[\hbar]$, where Λ_{Nov} is a suitable Novikov ring and $\hbar$ is the generator of $H^*(BS^1)$. He interprets this as being the S^1-equivariant Floer cohomology of X. Our construction of S^1-equivariant Floer homology provides an interpretation of the underlying homology group as the homology of a Floer-type complex. We expect that the D-module structure can also be defined within our setup.

Structure of the paper. In §2 we briefly recall the construction of symplectic homology. In §3 we introduce a new variant of it, which we call "parametrized symplectic homology". It corresponds to families of Hamiltonians, indexed by a finite dimensional parameter space. In order to assign a grading to the generators of the underlying chain complex, under the sole nondegeneracy assumption and regardless of the specific form of the Hamiltonian, we define the "parametrized Robbin-Salamon index" in §3.2. Section 4

is devoted to the S^1-equivariant theory. We recall in §4.1 the Borel construction and its interpretation in Morse homology. We define S^1-equivariant symplectic homology in §4.2, following Viterbo [27, §5]. We prove Theorems 1.1 and 1.2 in §5.1, using a Morse-Bott construction and a spectral sequence argument. In §5.2 we use similar techniques to study continuation maps. We prove in Appendix A some important properties of the parametrized Robbin-Salamon index.

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2 Symplectic homology

We briefly recall in this section the definition of symplectic homology, and we refer to [3] for full details. In the sequel (W, ω) denotes a compact symplectic manifold with contact type boundary $M := \partial W$. This means that there exists a vector field X defined in a neighbourhood of M, transverse and pointing outwards along M, and such that

$$\mathcal{L}_X \omega = \omega.$$

Such an X is called a **Liouville vector field**. The 1-form $\alpha := (\iota_X \omega)|_M$ is a contact form on M and is called the **Liouville 1-form**. We denote by $\xi := \ker \alpha$ the contact structure defined by α, and we note that the isotopy class of ξ is uniquely determined by ω. The **Reeb vector field** R_α is defined by the conditions $\ker\, \omega|_M = \langle R_\alpha \rangle$ and $\alpha(R_\alpha) = 1$. We denote by ϕ_α the flow of R_α. The **action spectrum** of (M, α) is defined by

$$\mathrm{Spec}(M, \alpha) := \{T \in \mathbb{R}^+ \mid \text{ there is a closed } R_\alpha\text{-orbit of period } T\}.$$

Let ϕ be the flow of X. We parametrize a neighbourhood U of M by

$$G : M \times [-\delta, 0] \to U, \qquad (p, t) \mapsto \phi^t(p).$$

Then $d(e^t\alpha)$ is a symplectic form on $M \times \mathbb{R}^+$ and G satisfies $G^*\omega = d(e^t\alpha)$. We denote by

$$\widehat{W} := W \bigcup_G M \times \mathbb{R}^+$$

the **symplectic completion of $\boldsymbol{W}$** and endow it with the symplectic form

$$\widehat{\omega} := \begin{cases} \omega & \text{on } W, \\ d(e^t\alpha) & \text{on } M \times \mathbb{R}^+. \end{cases}$$

Given a time-dependent Hamiltonian $H : S^1 \times \widehat{W} \to \mathbb{R}$ we define the **Hamiltonian vector field** X_H^θ by

$$\widehat{\omega}(X_H^\theta, \cdot) = dH_\theta, \qquad \theta \in S^1 = \mathbb{R}/\mathbb{Z},$$

where $H_\theta := H(\theta, \cdot)$. We denote by ϕ_H the flow of X_H^θ, defined by $\phi_H^0 = \mathrm{Id}$ and

$$\frac{d}{d\theta}\phi_H^\theta(x) = X_H^\theta(\phi_H^\theta(x)), \qquad \theta \in \mathbb{R}.$$

We denote by $\mathcal{P}(H)$ the set of 1-periodic orbits of X_H^θ, and we denote by $\mathcal{P}^a(H) \subset \mathcal{P}(H)$ the set of 1-periodic orbits in the free homotopy class a.

We define the class $\mathcal{H}$ of **admissible Hamiltonians** to consist of smooth functions $H : S^1 \times \widehat{W} \to \mathbb{R}$ satisfying the following conditions:

- $H < 0$ on W;
- there exists $t_0 \geq 0$ such that $H(\theta, p, t) = \beta e^t + \beta'$ for $t \geq t_0$, with $0 < \beta \notin \mathrm{Spec}(M, \alpha)$ and $\beta' \in \mathbb{R}$.

We denote by $\mathcal{H}_{\mathrm{reg}} \subset \mathcal{H}$ the dense set of Hamiltonians H such that all elements of $\mathcal{P}(H)$ are nondegenerate, i.e. the Poincaré return map has no eigenvalues equal to 1. Let a be a free homotopy class of loops in W. The **symplectic homology groups** of (W, ω) are defined by

$$SH_*^a(W, \omega) := \varinjlim_{H \in \mathcal{H}_{\mathrm{reg}}} SH_*^a(H, J).$$

Here J is an almost complex structure on $\widehat{W}$ which is compatible with $\widehat{\omega}$, convex and invariant under translation in the t-variable outside a compact set, and regular for H (in particular one must allow J to depend on θ). We denote by $SH_*^a(H, J)$ the Floer homology groups of the pair (H, J) in the free homotopy class a and with coefficients in the Novikov ring Λ_ω. We assume throughout this paper that W satisfies condition (1.3), so that the energy of a Floer trajectory does not depend on its homology class, but only on its endpoints. We refer to [3] for the details of the construction and in

particular for the definition of the coefficient ring Λ_ω. Throughout this paper the Novikov ring is understood to be defined over $\mathbb{Q}$.

For the trivial homotopy class $a = 0$ we denote the symplectic homology groups by $SH_*(W, \omega)$. The **reduced Hamiltonian action functional** is

$$\mathcal{A}^0_H : C^\infty_{\mathrm{contr}}(S^1, \widehat{W}) \to \mathbb{R},$$
$$\mathcal{A}^0_H(\gamma) := -\int_{D^2} \sigma^*\widehat{\omega} - \int_{S^1} H(\theta, \gamma(\theta))\, d\theta.$$

Here $C^\infty_{\mathrm{contr}}(S^1, \widehat{W})$ denotes the space of smooth contractible loops in $\widehat{W}$ and $\sigma : D^2 \to \widehat{W}$ is a smooth extension of γ. Note that $\mathcal{A}^0_H$ is well-defined thanks to condition (1.3) and is decreasing along Floer trajectories.

We now consider a special cofinal class of Hamiltonians $\mathcal{H}' \subset \mathcal{H}$, consisting of elements $H \in \mathcal{H}'$ which satisfy the following conditions:

- there exists $t_0 \geq 0$ such that $H(\theta, p, t) = \beta e^t + \beta'$ for $t \geq t_0$, with $0 < \beta \notin \mathrm{Spec}(M, \alpha)$ and $\beta' \in \mathbb{R}$;
- $H < 0$ and C^2-small on W;
- $H(\theta, p, t)$ is C^2-close to an increasing function of t on $S^1 \times M \times [0, t_0]$.

The last condition implies that, in the region $M \times [0, t_0]$, each 1-periodic orbit of H is C^1-close to a closed characteristic on some level $M \times \{t\}$.

Given $H \in \mathcal{H}'_{\mathrm{reg}} := \mathcal{H}_{\mathrm{reg}} \cap \mathcal{H}'$, a regular almost complex structure J, and a choice of $\epsilon > 0$ small enough, we define the chain complexes

$$SC^-_*(H, J) := \bigoplus_{\substack{\gamma \in \mathcal{P}^0(H) \\ \mathcal{A}^0_H(\gamma) \leq \epsilon}} \Lambda_\omega \langle \gamma \rangle \ \subset SC_*(H, J) \tag{2.1}$$

and

$$SC^+_*(H, J) := SC_*(H, J)/SC^-_*(H, J).$$

The differential on $SC^\pm_*(H, J)$ is induced by ∂. The groups

$$SH^\pm_*(H, J) := H_*(SC^\pm_*(H), \partial)$$

do not depend on J, nor on ϵ, and we define

$$SH^\pm_*(W, \omega) := \lim_{\substack{\longrightarrow \\ H \in \mathcal{H}'_{\mathrm{reg}}}} SH^\pm_*(H).$$

We call $SH^+_*(W, \omega)$ the **positive symplectic homology group** of (W, ω).

Remark 2.1. Condition (1.3) can be replaced in the case of contractible orbits by the weaker **symplectic asphericity** condition $\langle \omega, \pi_2(W) \rangle = 0$.

Let us assume now that W has **positive contact type** boundary [19, §5.4]. This means that every positively oriented closed characteristic γ on M which is contractible in W has positive action $\mathcal{A}_\omega(\gamma)$ bounded away from zero, where

$$\mathcal{A}_\omega(\gamma) := \int_{D^2} \sigma^*\omega$$

for some extension $\sigma : D^2 \to W$ of γ. This condition is automatically satisfied if the boundary M is of restricted contact type, i.e. the vector field X is globally defined on W. Under the positive contact type assumption we have [27]

$$SH_*^-(W, \omega) = H_{*+n}(W, \partial W; \Lambda_\omega), \qquad n = \frac{1}{2} \dim W,$$

and the short exact sequence of complexes $SC_*^-(H) \to SC_*(H) \to SC_*^+(H)$ induces the tautological long exact sequence (1.5).

3 Parametrized symplectic homology

We introduce in this section a new variant of Floer homology, which we call "parametrized Floer homology". In the sequel Λ is a finite dimensional closed manifold of dimension m, which we call "parameter space". The elements of Λ are denoted by λ. When the parameter space is S^{2N+1}, the parametrized symplectic homology groups will be the abutment of the spectral sequence which gives rise to the Gysin exact sequence (1.4).

3.1 The parametrized Floer equation

For each free homotopy class a in W, we fix a reference loop $l_a : S^1 \to \widehat{W}$ such that $[l_a] = a$. If a is the trivial homotopy class, we choose l_a to be a constant loop. Recall that free homotopy classes of loops in $\widehat{W}$ are in one-to-one correspondence with conjugacy classes in $\pi_1(\widehat{W})$. As a consequence, the inverse a^{-1} of a free homotopy class is well-defined. We require that $l_{a^{-1}}$ coincides with the loop l_a with the opposite orientation.

We define the set $\mathcal{H}_\Lambda$ of **admissible Hamiltonian families** to consist of elements $H \in C^\infty(S^1 \times \widehat{W} \times \Lambda, \mathbb{R})$ which satisfy the following conditions:

- $H < 0$ on $S^1 \times W \times \Lambda$;
- there exists $t_0 \geq 0$ such that $H(\theta, p, t, \lambda) = \beta e^t + \beta'(\lambda)$ for $t \geq t_0$, with $0 < \beta \notin \mathrm{Spec}(M, \alpha)$ and $\beta' \in C^\infty(\Lambda, \mathbb{R})$.

Let $H : S^1 \times \widehat{W} \times \Lambda \to \mathbb{R}$ be an admissible Hamiltonian family denoted by $H(\theta, x, \lambda) = H_\lambda(\theta, x)$. This defines a family of action functionals

$$\mathcal{A} : C^\infty(S^1, \widehat{W}) \times \Lambda \to \mathbb{R},$$

$$\mathcal{A}(\gamma, \lambda) = \mathcal{A}_\lambda(\gamma) := -\int_{[0,1]\times S^1} \sigma^*\omega - \int_{S^1} H_\lambda(\theta, \gamma(\theta))d\theta,$$

where $\sigma : [0,1]\times S^1 \to \widehat{W}$ is a smooth homotopy from $l_{[\gamma]}$ to γ. The functional $\mathcal{A}$ is well-defined due to our standing assumption (1.3).

The differential of $\mathcal{A}$ is given by

$$d\mathcal{A}(\gamma, \lambda) \cdot (\zeta, \ell) = \int_{S^1} \omega(\dot\gamma(\theta) - X_{H_\lambda}(\gamma(\theta)), \zeta(\theta))d\theta - \int_{S^1} \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)d\theta \cdot \ell \tag{3.1}$$

and therefore (γ, λ) is a critical point of $\mathcal{A}$ if and only if

$$\gamma \in \mathcal{P}(H_\lambda) \quad \text{and} \quad \int_{S^1} \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)\, d\theta = 0. \tag{3.2}$$

We denote by $\mathcal{P}(H)$ the set of critical points of $\mathcal{A}$ consisting of pairs (γ, λ) satisfying (3.2). We denote by $\mathcal{P}^a(H)$ the set of pairs $(\gamma, \lambda) \in \mathcal{P}(H)$ such that γ lies in the free homotopy class a.

Remark 3.1. Equation (3.2) can be interpreted as follows. Every loop $\gamma : S^1 \to \widehat{W}$ determines a function

$$F_\gamma : \Lambda \to \mathbb{R}, \qquad \lambda \mapsto \int_{S^1} H(\theta, \gamma(\theta), \lambda)\, d\theta. \tag{3.3}$$

A pair (γ, λ) belongs therefore to $\mathcal{P}(H)$ if and only if

$$\gamma \in \mathcal{P}(H_\lambda) \quad \text{and} \quad \lambda \in \mathrm{Crit}(F_\gamma).$$

Let $J = (J_\lambda^\theta)$, $\lambda \in \Lambda$, $\theta \in S^1$ be a family of θ-dependent compatible almost complex structures on $\widehat{W}$ which, at infinity, are invariant under translations in the t-variable and satisfy the relations

$$J_\lambda^\theta \xi = \xi, \qquad J_\lambda^\theta(\frac{\partial}{\partial t}) = R_\alpha. \tag{3.4}$$

Such an **admissible family of almost complex structures** J induces a family of L^2-metrics on the space $C^\infty(S^1, \widehat{W})$, parametrized by Λ and defined by

$$\langle \zeta, \eta \rangle_\lambda := \int_{S^1} \omega(\zeta(\theta), J_\lambda^\theta \eta(\theta)) d\theta, \quad \zeta, \eta \in T_\gamma C^\infty(S^1, \widehat{W}) = \Gamma(\gamma^* T\widehat{W}).$$

Such a metric can be coupled with any metric g on Λ and gives rise to a metric on $C^\infty(S^1, \widehat{W}) \times \Lambda$ acting at a point (γ, λ) by

$$\langle (\zeta, \ell), (\eta, k) \rangle_{J,g} := \langle \zeta, \eta \rangle_\lambda + g(\ell, k), \qquad (\zeta, \ell), (\eta, k) \in \Gamma(\gamma^* T\widehat{W}) \oplus T_\lambda \Lambda.$$

We denote by $\mathcal{J}_\Lambda$ the set of pairs (J, g) consisting of an admissible almost complex structure J on $\widehat{W}$ and of a Riemannian metric g on Λ.

The **parametrized Floer equation** is the gradient equation for $\mathcal{A}$ with respect to such a metric $\langle \cdot, \cdot \rangle_{J,g}$. More precisely, given $\overline{p} := (\overline{\gamma}, \overline{\lambda}), \underline{p} := (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ we denote by

$$\widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)$$

the **space of parametrized Floer trajectories**, consisting of pairs (u, λ) with

$$u : \mathbb{R} \times S^1 \to \widehat{W}, \qquad \lambda : \mathbb{R} \to \Lambda,$$

satisfying

$$\begin{aligned} \partial_s u + J_{\lambda(s)}^\theta (\partial_\theta u - X_{H_{\lambda(s)}}^\theta(u)) &= 0, &(3.5)\\ \dot{\lambda}(s) - \int_{S^1} \vec{\nabla}_\lambda H(\theta, u(s,\theta), \lambda(s)) d\theta &= 0, &(3.6) \end{aligned}$$

and

$$\lim_{s \to -\infty} (u(s, \cdot), \lambda(s)) = (\overline{\gamma}, \overline{\lambda}), \quad \lim_{s \to +\infty} (u(s, \cdot), \lambda(s)) = (\underline{\gamma}, \underline{\lambda}). \tag{3.7}$$

Here and in the sequel we use the notation $\vec{\nabla}$ for a gradient vector field, whereas ∇ will denote a covariant derivative.

Remark 3.2. Equation (3.6) is equivalent to

$$\dot{\lambda}(s) - \vec{\nabla} F_{u(s,\cdot)}(\lambda(s)) = 0, \tag{3.8}$$

where $F_{u(s,\cdot)}$ is defined by (3.3). Thus, the parametrized Floer equation is a system involving a Floer equation and a finite-dimensional gradient equation.

The additive group $\mathbb{R}$ acts on $\widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)$ by reparametrization in the s-variable and we denote by

$$\mathcal{M}(\overline{p}, \underline{p}; H, J, g) := \widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)/\mathbb{R}$$

the **moduli space of parametrized Floer trajectories**.

Let us fix $p \geq 2$. The linearization of the equations (3.5-3.6) gives rise to the operator

$$D_{(u,\lambda)} : W^{1,p}(u^*T\widehat{W}) \oplus W^{1,p}(\lambda^*T\Lambda) \to L^p(u^*T\widehat{W}) \oplus L^p(\lambda^*T\Lambda),$$

$$D_{(u,\lambda)}(\zeta, \ell) := \begin{pmatrix} D_u\zeta + (D_\lambda J \cdot \ell)(\partial_\theta u - X_{H_\lambda}(u)) - J_\lambda(D_\lambda X_{H_\lambda} \cdot \ell) \\ \nabla_s \ell - \nabla_\ell \int_{S^1} \vec{\nabla}_\lambda H(\theta, u, \lambda) d\theta - \int_{S^1} \nabla_\zeta \vec{\nabla}_\lambda H(\theta, u, \lambda) d\theta \end{pmatrix},$$

where

$$D_u : W^{1,p}(u^*T\widehat{W}) \to L^p(u^*T\widehat{W})$$

is the usual Floer operator given by

$$D_u\zeta := \nabla_s\zeta + J_\lambda \nabla_\theta \zeta - J_\lambda \nabla_\zeta X_{H_\lambda} + \nabla_\zeta J_\lambda(\partial_\theta u - X_{H_\lambda}).$$

The Hessian of $\mathcal{A}$ at a critical point $p = (\gamma, \lambda)$ is given by the formula

$$\begin{aligned} & d^2\mathcal{A}(\gamma, \lambda)\big((\zeta, \ell), (\eta, k)\big) \\ = \;& \int_{S^1} \omega(\nabla_\theta \eta - \nabla_\eta X_{H_\lambda}, \zeta) d\theta - \int_{S^1} \eta(\frac{\partial H}{\partial \lambda} \cdot \ell) d\theta \\ & - \int_{S^1} k(dH_\lambda \cdot \zeta) d\theta - \int_{S^1} \frac{\partial^2 H}{\partial \lambda^2}(\ell, k) d\theta \\ = \;& d^2\mathcal{A}_{H_\lambda}(\gamma)(\zeta, \eta) - \int_{S^1} \eta(\frac{\partial H}{\partial \lambda} \cdot \ell) d\theta - \int_{S^1} k(dH_\lambda \cdot \zeta) d\theta - d^2F_\gamma(\lambda)(\ell, k). \end{aligned} \tag{3.9}$$

We define the asymptotic operator at a critical point (γ, λ) by

$$D_{(\gamma,\lambda)} : H^1(S^1, \gamma^*T\widehat{W}) \times T_\lambda\Lambda \to L^2(S^1, \gamma^*T\widehat{W}) \times T_\lambda\Lambda,$$

$$D_{(\gamma,\lambda)}(\zeta,\ell) = \begin{pmatrix} J_\lambda(\nabla_\theta\zeta - \nabla_\zeta X_{H_\lambda} - (D_\lambda X_{H_\lambda})\cdot \ell) \\ -\int_{S^1}\nabla_\zeta \frac{\partial H}{\partial\lambda} d\theta - \int_{S^1}\nabla_\ell \frac{\partial H}{\partial \lambda} d\theta \end{pmatrix}. \tag{3.10}$$

Note that $D_{(\gamma,\lambda)}$ is obtained from $D_{(u,\lambda)}$ for $(u(s,\theta),\lambda(s)) \equiv (\gamma(\theta),\lambda)$ and $(\zeta(s,\theta),\ell(s)) \equiv (\zeta(\theta),\ell)$.

We say that a critical point (γ,λ) is **nondegenerate** if the Hessian $d^2\mathcal{A}(\gamma,\lambda)$ has trivial kernel. In [5, Lemma 2.3] we proved that this condition is equivalent to the injectivity of the asymptotic operator $D_{(\gamma,\lambda)}$. Since the latter is self-adjoint, this condition is also equivalent to its surjectivity.

Remark 3.3. We note that nondegeneracy of a critical point (γ,λ) does not imply that γ is a nondegenerate orbit of H_λ, nor that λ is a nondegenerate critical point of F_γ. This situation is already present in Morse theory, as the following example shows. We consider the Morse function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $(x,y)\mapsto xy$. Then $(x_0,y_0)=(0,0)$ is a nondegenerate critical point, but the restrictions of f to $\mathbb{R}\times\{0\}$ and $\{0\}\times\mathbb{R}$ are constant, hence $x_0=0$ and $y_0=0$ are degenerate critical points.

An admissible Hamiltonian family H is called **nondegenerate** if $\mathcal{P}(H)$ consists of nondegenerate elements. We denote the set of nondegenerate and admissible Hamiltonian families by $\mathcal{H}_{\Lambda,\mathrm{reg}} \subset \mathcal{H}_\Lambda$. By [5, Proposition 2.5], the set $\mathcal{H}_{\Lambda,\mathrm{reg}}$ is of the second Baire category in $\mathcal{H}_\Lambda$. Moreover, if $H\in\mathcal{H}_{\Lambda,\mathrm{reg}}$ the set $\mathcal{P}(H)$ is discrete.

We denote

$$\begin{aligned} \mathcal{W}^{1,p} &:= W^{1,p}(\mathbb{R}\times S^1, u^*T\widehat{W}) \oplus W^{1,p}(\mathbb{R},\lambda^*T\Lambda), \\ \mathcal{L}^p &:= L^p(\mathbb{R}\times S^1, u^*T\widehat{W}) \oplus L^p(\mathbb{R},\lambda^*T\Lambda). \end{aligned}$$

Let $(\overline{\gamma},\overline{\lambda}),(\underline{\gamma},\underline{\lambda})\in\mathcal{P}(H)$ be nondegenerate. We proved in [5, Theorem 2.6] that, given any $(u,\lambda)\in\widehat{\mathcal{M}}((\overline{\gamma},\overline{\lambda}),(\underline{\gamma},\underline{\lambda});H,J,g)$, the operator

$$D_{(u,\lambda)} : \mathcal{W}^{1,p} \to \mathcal{L}^p$$

is Fredholm for $1<p<\infty$.

Remark 3.4. We can choose a unitary trivialization of $u^*T\widehat{W}$ and a trivialization of $\lambda^*T\Lambda$ in which $D_{(u,\lambda)}$ has the form

$$D_{(u,\lambda)}\begin{pmatrix}\zeta\\ \ell\end{pmatrix} := \left[\begin{pmatrix}\partial_s + J_0\partial_\theta & 0 \\ 0 & \frac{d}{ds}\end{pmatrix} + N\right]\begin{pmatrix}\zeta\\ \ell\end{pmatrix}, \tag{3.11}$$

with $N : \mathbb{R} \times S^1 \to \mathrm{Mat}_{2n+m}(\mathbb{R})$ pointwise bounded and $\lim_{s\to\pm\infty} N(s,\theta)$ symmetric.

Let $H \in \mathcal{H}_{\Lambda,\mathrm{reg}}$. A pair $(J,g) \in \mathcal{J}_\Lambda$ is called **regular for $\boldsymbol{H}$** if the operator $D_{(u,\lambda)}$ is surjective for any solution (u,λ) of (3.5-3.7). We denote the space of such pairs by $\mathcal{J}_{\Lambda,\mathrm{reg}}(H)$. We proved in [5, Theorem 4.1] that there exists a subset of second Baire category $\mathcal{HJ}_{\Lambda,\mathrm{reg}} \subset \mathcal{H}_{\Lambda,\mathrm{reg}} \times \mathcal{J}_\Lambda$ such that $H \in \mathcal{H}_{\Lambda,\mathrm{reg}}$ and $(J,g) \in \mathcal{J}_{\Lambda,\mathrm{reg}}(H)$ whenever $(H,J,g) \in \mathcal{HJ}_{\Lambda,\mathrm{reg}}$.

As a consequence, whenever $(H,J,g) \in \mathcal{HJ}_{\Lambda,\mathrm{reg}}$, the moduli spaces of parametrized Floer trajectories $\mathcal{M}(\overline{p},\underline{p};H,J,g)$ are smooth manifolds, for all $\overline{p},\underline{p} \in \mathcal{P}(H)$. The local dimension at $(u,\lambda) \in \mathcal{M}(\overline{p},\underline{p};H,J,g)$ is equal to $\mathrm{ind}\, D_{(u,\lambda)} - 1$. The purpose of the next section is to compute this Fredholm index.

3.2 The parametrized Robbin-Salamon index

Recall that, for each free homotopy class a in $\widehat{W}$, we have chosen in Section 3.1 a reference loop l_a such that $[l_a] = a$. We now choose a symplectic trivialization
$$\Phi_a^1 : S^1 \times \mathbb{R}^{2n} \to l_a^* T\widehat{W}$$
for each free homotopy class a. If a is the trivial homotopy class we choose the trivialization to be constant.

For each $p = (\gamma,\lambda) \in \mathcal{P}(H)$ we choose a smooth homotopy $\sigma_p : [0,1] \times S^1 \to \widehat{W}$ such that $\sigma_p(0,\cdot) = l_{[\gamma]}$ and $\sigma_p(1,\cdot) = \gamma$. This gives rise to a unique (up to homotopy) symplectic trivialization
$$\Phi_p^1 : [0,1] \times S^1 \times \mathbb{R}^{2n} \to \sigma_p^* T\widehat{W}$$
such that $\Phi_p^1 = \Phi_{[\gamma]}^1$ on $\{0\} \times S^1 \times \mathbb{R}^{2n}$. Moreover, we fix an isometry $\Phi_p^2 : \mathbb{R}^m \to T_\lambda \Lambda$.

We define the subgroup $\mathcal{S}_{n,m} \subset \mathrm{Sp}(2n+2m)$ to consist of matrices of the form
$$M = M(\Psi,X,E) = \begin{pmatrix} \Psi & \Psi X & 0 \\ 0 & \mathbb{1} & 0 \\ X^T J_0 & E + \frac{1}{2} X^T J_0 X & \mathbb{1} \end{pmatrix},$$
with $\Psi \in \mathrm{Sp}(2n)$, $X \in \mathrm{Mat}_{2n,m}(\mathbb{R})$, and $E \in \mathrm{Mat}_m(\mathbb{R})$ symmetric. Here we have denoted $J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ the standard complex structure on $\mathbb{R}^{2n}$, and

the elements $\Psi \in \mathrm{Sp}(2n)$ are characterized by the condition $\Psi^T J_0 \Psi = J_0$. Similarly, we denote the standard complex structure on $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$ by

$$\widetilde{J}_0 := \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & -\mathbb{1} \\ 0 & \mathbb{1} & 0 \end{pmatrix},$$

and the elements $\widetilde{\Psi} \in \mathrm{Sp}(2n+2m)$ are characterized by the condition $\widetilde{\Psi}^T \widetilde{J}_0 \widetilde{\Psi} = \widetilde{J}_0$. That $\mathcal{S}_{n,m}$ is a subgroup follows from the relations

$$\begin{aligned} & M(\Psi_1, X_1, E_1) \cdot M(\Psi_2, X_2, E_2) \\ & \quad = \; M(\Psi_1\Psi_2, X_2 + \Psi_2^{-1} X_1, E_1 + E_2 + \mathrm{Sym}(X_1^T J_0 \Psi_2 X_2)) \end{aligned}$$

and

$$M(\Psi, X, E)^{-1} = M(\Psi^{-1}, -\Psi X, -E). \tag{3.12}$$

Here we have denoted by

$$\mathrm{Sym}(P) := (P^T + P)/2$$

the symmetric part of a square matrix P.

To an element $p = (\gamma, \lambda) \in \mathcal{P}(H)$ equipped with a unitary trivialization of $\gamma^* T\widehat{W}$ and an isometry $T_\lambda \Lambda \equiv \mathbb{R}^m$, we will now associate a path

$$M(\theta) = M(\Psi(\theta), X(\theta), E(\theta)), \qquad \theta \in [0,1],$$

with $M(0) = \mathbb{1} = M(\mathbb{1}, 0, 0)$. In the given trivialization of $T(\widehat{W} \times \Lambda)$ along γ, the linearization of the flow Φ^θ of the Hamiltonian vector field X_H^θ has the form

$$\begin{aligned} T_{(\gamma(0),\lambda)}(\widehat{W} \times \Lambda) \;&\to\; T_{(\gamma(\theta),\lambda)}(\widehat{W} \times \Lambda), \\ (\zeta_0, l) \;&\mapsto\; (\Psi(\theta)\zeta_0 + \Psi(\theta)X(\theta)l, l). \end{aligned} \tag{3.13}$$

This uniquely defines the matrices $\Psi(\theta)$ and $X(\theta)$. The matrices $E(\theta)$ are defined to be the symmetric part of the endomorphisms

$$\begin{aligned} T_\lambda\Lambda \;&\to\; T_\lambda\Lambda, \\ l \;&\mapsto\; -\frac{d}{d\lambda}\int_0^\theta \vec{\nabla}_\lambda H(\tau, \Phi^\tau(\gamma(0), \lambda), \lambda)\, d\tau \cdot l. \end{aligned} \tag{3.14}$$

We define the **parametrized Robbin-Salamon index** $\mu(p)$ of p with respect to the given trivialization as the Robbin-Salamon index [20] of the path $M : [0,1] \to \mathcal{S}_{n,m} \subset \mathrm{Sp}(2n+2m)$:

$$\mu(p) := \mu_{RS}(M(\cdot)). \tag{3.15}$$

Note that the path M corresponds to the linearized Hamiltonian flow of $\widetilde{H} : S^1 \times \widehat{W} \times T^*\Lambda \to \mathbb{R}$ defined by $\widetilde{H}(\theta, x, (\lambda, p)) := H(\theta, x, \lambda)$ (see Definition 1.6 in the Introduction). We refer to Appendix A for a summary of the properties of the parametrized Robbin-Salamon index.

Theorem 3.5. *Assume* $(\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ *are nondegenerate and fix* $1 < p < \infty$. *For any* $(u, \lambda) \in \widehat{\mathcal{M}}((\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}); H, J, g)$ *the index of the Fredholm operator* $D_{(u,\lambda)} : \mathcal{W}^{1,p} \to \mathcal{L}^p$ *is*

$$\mathrm{ind}\, D_{(u,\lambda)} = -\mu(\overline{\gamma}, \overline{\lambda}) + \mu(\underline{\gamma}, \underline{\lambda}).$$

In the above statement, it is understood that the trivialization used to define $\mu(\overline{\gamma}, \overline{\lambda})$ is obtained from the trivialization used to define $\mu(\underline{\gamma}, \underline{\lambda})$ by continuation along the map u. We denote by

$$\overline{M} := M(\overline{\Psi}, \overline{X}, \overline{E}), \qquad \underline{M} := M(\underline{\Psi}, \underline{X}, \underline{E})$$

the paths (based at $\mathbb{1}$) in $\mathcal{S}_{n,m}$ used to define $\mu(\overline{\gamma}, \overline{\lambda})$, respectively $\mu(\underline{\gamma}, \underline{\lambda})$.

Proof. By elliptic regularity for $D_{(u,\lambda)}$ (and its formal adjoint), it is enough to prove the statement for $p = 2$. Indeed, the kernel and cokernel of $D_{(u,\lambda)}$ are spanned by smooth elements, so that the index does not depend on p. Given a unitary trivialization of $u^*T\widehat{W}$ and an orthogonal trivialization of $\lambda^*T\Lambda$ we can write $D_{(u,\lambda)}$ as

$$D_{(u,\lambda)} : H^1(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times H^1(\mathbb{R}, \mathbb{R}^m) \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times L^2(\mathbb{R}, \mathbb{R}^m),$$

$$D_{(u,\lambda)}(\zeta, \ell) = \begin{pmatrix} \partial_s \zeta \\ \partial_s \ell \end{pmatrix} + A(s) \begin{pmatrix} \zeta \\ \ell \end{pmatrix},$$

where $A(s) : H^1(S^1, \mathbb{R}^{2n}) \times \mathbb{R}^m \to L^2(S^1, \mathbb{R}^{2n}) \times \mathbb{R}^m$ has the property that $A(s) \to A^\pm$, $s \to \pm\infty$ and $A^\pm$ coincide through the given trivializations with the asymptotic operators $D_{(\underline{\gamma}, \underline{\lambda})}$ and $D_{(\overline{\gamma}, \overline{\lambda})}$, which are bijective in view of our standing nondegeneracy assumption.

In order to compute the Fredholm index of the operator $D_{(u,\lambda)}$, we use the spectral flow of the family of self-adjoint operators $A(s)$, $s \in \mathbb{R}$ [21, Theorem A]. In view of (3.10), these operators can be written in the given trivializations of $T\widehat{W}$ and $T\Lambda$ along u and λ as

$$A(s)(\zeta, l) = \begin{pmatrix} J_0 \frac{\partial}{\partial\theta}\zeta(\theta) + S(s,\theta)\zeta(\theta) + C(s,\theta)^T l \\ \int_{S^1} C(s,\theta)\zeta(\theta)\, d\theta + \int_{S^1} D(s,\theta)\, d\theta\ l \end{pmatrix}, \tag{3.16}$$

where $S(s,\theta) = S(s,\theta)^T$ and $D(s,\theta) = D(s,\theta)^T$ are symmetric matrices.

Let us compute the kernel of the operator $A(s)$ for a fixed value of $s \in \mathbb{R}$. We define

$$\Psi : \mathbb{R} \times [0,1] \to \mathrm{Sp}(2n)$$

by $\dot{\Psi}(s,\theta) = J_0 S(s,\theta)\Psi(s,\theta)$ and $\Psi(s,0) = \mathbb{1}$, so that

$$\lim_{s\to-\infty} \Psi(s,\cdot) = \overline{\Psi}(\cdot), \qquad \lim_{s\to\infty} \Psi(s,\cdot) = \underline{\Psi}(\cdot).$$

For $(\zeta, l) \in \ker A(s)$, we write $\zeta(\theta) = \Psi(s,\theta)\eta(\theta)$ for some smooth function $\eta : [0,1] \to \mathbb{R}^{2n}$. Substituting this in the first component of $A(s)(\zeta, l)$, we obtain

$$\dot{\eta}(\theta) = \Psi(s,\theta)^{-1} J_0 C(s,\theta)^T l. \tag{3.17}$$

We define

$$X : \mathbb{R} \times [0,1] \to \mathrm{Mat}_{2n,m}(\mathbb{R})$$

by $\dot{X}(s,\theta) = \Psi(s,\theta)^{-1} J_0 C(s,\theta)^T$ and $X(s,0) = 0$. The solution of (3.17) is then $\eta(\theta) = X(s,\theta)l + \eta(0)$, so that

$$\zeta(\theta) = \Psi(s,\theta)\zeta_0 + \Psi(s,\theta)X(s,\theta)l, \tag{3.18}$$

with $\zeta_0 = \zeta(0) = \eta(0)$. Comparing (3.18) with (3.13) we see that

$$\lim_{s\to-\infty} X(s,\cdot) = \overline{X}(\cdot), \qquad \lim_{s\to\infty} X(s,\cdot) = \underline{X}(\cdot).$$

The solution $\zeta(\theta)$ given by (3.18) descends to $S^1 = \mathbb{R}/\mathbb{Z}$ if and only if

$$\zeta_0 = \Psi(s,1)\zeta_0 + \Psi(s,1)X(s,1)l. \tag{3.19}$$

Substituting the expression (3.18) for $\zeta(\theta)$ in the second component of $A(s)(\zeta, l)$, we obtain

$$\int_0^1 C(s,\theta)\Psi(s,\theta) d\theta\, \zeta_0 + \int_0^1 \big(C(s,\theta)\Psi(s,\theta)X(s,\theta) + D(s,\theta)\big) d\theta\, l = 0. \tag{3.20}$$

We now notice that, for any $\theta \in [0,1]$, we have

$$\int_0^\theta C(\tau)\Psi(\tau)\,d\tau = \int_0^\theta \dot{X}(\tau)^T \Psi(\tau)^T J_0 \Psi(\tau)\,d\tau = X(\theta)^T J_0. \tag{3.21}$$

We define

$$E : \mathbb{R} \times [0,1] \to \mathrm{Mat}_m(\mathbb{R})$$

by

$$E(s,\theta) = \int_0^\theta \big(C(s,\tau)\Psi(s,\tau)X(s,\tau) + D(s,\tau)\big)d\tau - \frac{1}{2}X(s,\theta)^T J_0 X(s,\theta). \tag{3.22}$$

We claim that $\frac{1}{2}X(s,\theta)^T J_0 X(s,\theta)$ is the anti-symmetric part of the matrix $\int_0^\theta C(s,\tau)\Psi(s,\tau)X(s,\tau)d\tau$, so that $E(s,\theta)$ is symmetric. Omitting the s-variable for clarity, and using that $C(\tau)\Psi(\tau) = \dot{X}(\tau)^T J_0$, we obtain

$$\begin{aligned} &\int_0^\theta C(\tau)\Psi(\tau)X(\tau)d\tau - \int_0^\theta X(\tau)^T \Psi(\tau)^T C(\tau)^T d\tau \\ &\qquad = \int_0^\theta \dot{X}(\tau)^T J_0 X(\tau)d\tau + \int_0^\theta X(\tau)^T J_0 \dot{X}(\tau)d\tau \\ &\qquad = X(\theta)^T J_0 X(\theta). \end{aligned}$$

It follows that $E(s,\theta)$ is the symmetric part of $\int_0^\theta \big(C(s,\tau)\Psi(s,\tau)X(s,\tau) + D(s,\tau)\big)d\tau$. Comparing this with (3.14), it follows that

$$\lim_{s\to-\infty} E(s,\cdot) = \overline{E}(\cdot), \qquad \lim_{s\to\infty} E(s,\cdot) = \underline{E}(\cdot).$$

With our new notations in place, we see that (3.19) and (3.20) are equivalent to the $(2n+m)\times(2n+m)$ system of linear equations

$$\begin{pmatrix} \Psi(s,1) - \mathbb{1} & \Psi(s,1)X(s,1) \\ X(s,1)^T J_0 & E(s,1) + \frac{1}{2}X(s,1)^T J_0 X(s,1) \end{pmatrix} \begin{pmatrix} \zeta_0 \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.23}$$

The solutions of the system (3.23) are in bijective correspondence with the elements $(\zeta, l) \in \ker A(s)$ through equation (3.18). On the other hand, it follows from the definition of $\mathcal{S}_{n,m}$ that solutions of (3.23) are in bijective correspondence with elements

$$(\zeta_0, l, 0) \in \ker\big(M(\Psi(s,1), X(s,1), E(s,1)) - \mathbb{1}\big).$$

Since $(0,0,v) \in \ker\left(M(\Psi(s,1),X(s,1),E(s,1)) - \mathbb{1}\right)$ for all $v \in \mathbb{R}^m$, we infer that $\ker A(s) \neq 0$ if and only if

$$\dim\ker\left(M(\Psi(s,1),X(s,1),E(s,1)) - \mathbb{1}\right) > m. \tag{3.24}$$

Remark 3.6. *To each operator $A(s)$ of the form* (3.16) *we associated a path of matrices $M : [0,1] \to \mathcal{S}_{n,m}$, $M(\theta) = M(\Psi(\theta),X(\theta),E(\theta))$ such that $M(0) = \mathbb{1}$. Conversely, any such path M determines a unique operator $A(s)$ of the form* (3.16) *by the formulas*

$$\begin{aligned} S(\theta) &= -J_0\dot{\Psi}(\theta)\Psi(\theta)^{-1} \\ C(\theta) &= \dot{X}(\theta)^T\Psi(\theta)^T J_0 \\ D(\theta) &= \dot{E}(\theta) + \mathrm{Sym}\left(X(\theta)^T J_0 \dot{X}(\theta)\right). \end{aligned}$$

We now compute the crossing form $\Gamma(A,s)$ on $\ker A(s)$ for the spectral flow of the family of operators $A(s)$, $s \in \mathbb{R}$. Recall that it is defined by $\Gamma(A,s)(\zeta,l) = \langle(\zeta,l), \frac{d}{ds}A(s)(\zeta,l)\rangle$ for all $(\zeta,l) \in \ker A(s)$. The operator $\frac{d}{ds}A(s)$ is given by

$$\frac{d}{ds}A(s)(\zeta,l) = \begin{pmatrix} \frac{\partial}{\partial s}S(s,\theta)\zeta(\theta) + \frac{\partial}{\partial s}C(s,\theta)^T l \\ \int_{S^1}\frac{\partial}{\partial s}C(s,\theta)\zeta(\theta)d\theta + \int_{S^1}\frac{\partial}{\partial s}D(s,\theta)d\theta\ l \end{pmatrix}.$$

Since $(\zeta,l) \in \ker A(s)$, we have $\zeta(\theta) = \Psi(s,\theta)\zeta_0 + \Psi(s,\theta)X(s,\theta)l$. We obtain

$$\begin{aligned} &\Gamma(A,s)(\zeta,l) \\ &= \int_{S^1}\left\langle \zeta(\theta), \frac{\partial}{\partial s}S(s,\theta)\zeta(\theta) + \frac{\partial}{\partial s}C(s,\theta)^T l\right\rangle d\theta \\ &\quad + \left\langle l, \int_{S^1}\frac{\partial}{\partial s}C(s,\theta)\zeta(\theta)\,d\theta + \int_{S^1}\frac{\partial}{\partial s}D(s,\theta)\,d\theta\ l\right\rangle \\ &= \int_0^1 (\zeta_0 + X(s,\theta)l)^T\Psi(s,\theta)^T\frac{\partial}{\partial s}S(s,\theta)\Psi(s,\theta)(\zeta_0 + X(s,\theta)l)\,d\theta \qquad (3.25) \\ &\quad + \int_0^1 (\zeta_0 + X(s,\theta)l)^T\Psi(s,\theta)^T\frac{\partial}{\partial s}C(s,\theta)^T l\,d\theta \\ &\quad + l^T\int_0^1 \frac{\partial}{\partial s}C(s,\theta)\Psi(s,\theta)(\zeta_0 + X(s,\theta)l)\,d\theta \\ &\quad + l^T\int_{S^1}\frac{\partial}{\partial s}D(s,\theta)d\theta\, l. \qquad (3.26) \end{aligned}$$

Let us define symmetric matrices $\widehat{S}(s,\theta)$ by $\frac{\partial}{\partial s}\Psi(s,\theta) = J_0\widehat{S}(s,\theta)\Psi(s,\theta)$. The condition $\Psi(s,0) = \mathbb{1}$ implies $\widehat{S}(s,0) = 0$. We claim that (see also [22, proof of Lemma 2.6])

$$\Psi(s,\theta)^T \frac{\partial}{\partial s} S(s,\theta)\Psi(s,\theta) = \frac{\partial}{\partial \theta}\left(\Psi(s,\theta)^T \widehat{S}(s,\theta)\Psi(s,\theta)\right). \tag{3.27}$$

Dropping the (s,θ) variables for clarity, we have [22]

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\Psi^T\widehat{S}\Psi\right) &= \Psi^T S^T(-J_0)\widehat{S}\Psi + \Psi^T\frac{\partial}{\partial\theta}(\widehat{S}\Psi) \\
&= -\Psi^T S\frac{\partial}{\partial s}\Psi + \Psi^T\frac{\partial}{\partial\theta}\left(-J_0\frac{\partial}{\partial s}\Psi\right) \\
&= -\Psi^T S\frac{\partial}{\partial s}\Psi - \Psi^T J_0\frac{\partial}{\partial s}\frac{\partial}{\partial\theta}\Psi \\
&= -\Psi^T S\frac{\partial}{\partial s}\Psi - \Psi^T J_0\frac{\partial}{\partial s}(J_0 S\Psi) \\
&= \Psi^T\frac{\partial}{\partial s}S\Psi.
\end{aligned}$$

Using (3.27), the term (3.25) becomes

$$\begin{aligned}
&\int_0^1 (\zeta_0 + X(s,\theta)l)^T\frac{\partial}{\partial\theta}\left(\Psi(s,\theta)^T\widehat{S}(s,\theta)\Psi(s,\theta)\right)(\zeta_0 + X(s,\theta)l)\,d\theta \\
&= (\zeta_0 + X(s,1)l)^T\Psi(s,1)^T\widehat{S}(s,1)\Psi(s,1)(\zeta_0 + X(s,1)l) \\
&\quad -l^T\int_0^1\frac{\partial}{\partial\theta}X(s,\theta)^T\Psi(s,\theta)^T\widehat{S}(s,\theta)\Psi(s,\theta)(\zeta_0 + X(s,\theta)l)\,d\theta \\
&\quad -\int_0^1(\zeta_0 + X(s,\theta)l)^T\Psi(s,\theta)^T\widehat{S}(s,\theta)\Psi(s,\theta)\frac{\partial}{\partial\theta}X(s,\theta)\,d\theta\ l \\
&= \zeta_0^T\widehat{S}(s,1)\zeta_0 + l^T\int_0^1 C(s,\theta)J_0\widehat{S}(s,\theta)\Psi(s,\theta)(\zeta_0 + X(s,\theta)l)\,d\theta \\
&\quad -\int_0^1(\zeta_0 + X(s,\theta)l)^T\Psi(s,\theta)^T\widehat{S}(s,\theta)J_0C(s,\theta)^T\,d\theta\ l \\
&= \zeta_0^T\widehat{S}(s,1)\zeta_0 + l^T\int_0^1 C(s,\theta)\frac{\partial}{\partial s}\Psi(s,\theta)(\zeta_0 + X(s,\theta)l)\,d\theta \\
&\quad +\int_0^1(\zeta_0 + X(s,\theta)l)^T\frac{\partial}{\partial s}\Psi(s,\theta)^T C(s,\theta)^T\,d\theta\ l.
\end{aligned}$$

The second equality uses (3.19). Thus, equation (3.26) becomes

$$\begin{aligned}
\Gamma(A,s)(\zeta,l) &= \zeta_0^T \widehat{S}(s,1)\zeta_0 + l^T \int_0^1 \frac{\partial}{\partial s}\big(C(s,\theta)\Psi(s,\theta)\big)(\zeta_0 + X(s,\theta)l)d\theta \\
&\quad + \int_0^1 (\zeta_0 + X(s,\theta)l)^T \frac{\partial}{\partial s}\big(\Psi(s,\theta)^T C(s,\theta)^T\big) d\theta\, l \\
&\quad + l^T \int_{S^1} \frac{\partial}{\partial s} D(s,\theta) d\theta\, l.
\end{aligned}$$

We claim that the matrix of the quadratic form $\Gamma(A,s)$ acting on the vector space of elements (ζ_0, l) satisfying (3.19) is given by

$$\begin{pmatrix} \widehat{S}(s,1) & -J_0 \frac{\partial}{\partial s} X(s,1) \\ \frac{\partial}{\partial s} X(s,1)^T J_0 & \frac{\partial}{\partial s} E(s,1) - \mathrm{Sym}\big(X^T(s,1) J_0 \frac{\partial}{\partial s} X(s,1)\big) \end{pmatrix}. \tag{3.28}$$

For the anti-diagonal terms, the claim follows from (3.21). For the term in the lower right corner, we compute

$$\begin{aligned}
\frac{\partial}{\partial s} E(s,1) &= \int_0^1 \frac{\partial}{\partial s} D(s,\theta) + \frac{\partial}{\partial s}\mathrm{Sym}(C(s,\theta)\Psi(s,\theta)X(s,\theta))\, d\theta \\
&= \int_0^1 \frac{\partial}{\partial s} D(s,\theta) + \mathrm{Sym}\big(\frac{\partial}{\partial s}(C(s,\theta)\Psi(s,\theta))X(s,\theta)\big)\, d\theta \\
&\qquad + \int_0^1 \mathrm{Sym}\big(C(s,\theta)\Psi(s,\theta)\frac{\partial}{\partial s} X(s,\theta)\big)\, d\theta, \quad (3.29)
\end{aligned}$$

while, in view of $C\Psi = \dot{X}^T J_0$, the term (3.29) becomes

$$\begin{aligned}
&\int_0^1 \mathrm{Sym}\big(\dot{X}(s,\theta)^T J_0 \frac{\partial}{\partial s} X(s,\theta)\big)\, d\theta \\
&\quad = \mathrm{Sym}\big(X(s,1)^T J_0 \frac{\partial}{\partial s} X(s,1)\big) - \int_0^1 \mathrm{Sym}\big(X(s,\theta)^T J_0 \frac{\partial}{\partial s} \dot{X}(s,\theta)\big)\, d\theta \\
&\quad = \mathrm{Sym}\big(X(s,1)^T J_0 \frac{\partial}{\partial s} X(s,1)\big) + \int_0^1 \mathrm{Sym}\big(X(s,\theta)^T \frac{\partial}{\partial s}(\Psi(s,\theta)^T C(s,\theta)^T)\big)\, d\theta.
\end{aligned}$$

Let us now compute the crossing form $\Gamma(M,s)$ for the Robbin-Salamon index of the path

$$s \mapsto M(s,1) = M(\Psi(s,1), X(s,1), E(s,1)).$$

By definition, the crossing form is $\Gamma(M,s)(\zeta_0,l,v)=\langle(\zeta_0,l,v),Q(s)(\zeta_0,l,v)\rangle$, with $Q(s):=-\widetilde{J}_0\frac{\partial}{\partial s}M(s,1)M(s,1)^{-1}$. Using (3.12) and the definition of $\widehat{S}(s,1)=-J_0\frac{\partial}{\partial s}\Psi(s,1)\Psi(s,1)^{-1}$, a straightforward computation shows that $Q(s)$ is given by

$$\begin{pmatrix} \widehat{S}(s,1) & -J_0\Psi(s,1)\frac{\partial}{\partial s}X(s,1) & 0 \\ \frac{\partial}{\partial s}X(s,1)^T\Psi(s,1)^TJ_0 & \frac{\partial}{\partial s}E(s,1)+\mathrm{Sym}\big(X(s,1)^TJ_0\frac{\partial}{\partial s}X(s,1)\big) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The key observation now is that, for any $(\zeta_0,l,0)\in\ker(M(s,1)-\mathbb{1})$, we have

$$\Gamma(M,s)(\zeta_0,l,0)=\Gamma(A,s)(\zeta,l),$$

with $\zeta(\theta)=\Psi(s,\theta)\zeta_0+\Psi(s,\theta)X(s,\theta)l$. This is seen by a straightforward computation, substituting $\zeta_0=\Psi(s,1)\zeta_0+\Psi(s,1)X(s,1)l$ in the non-diagonal terms of $\Gamma(M,s)(\zeta_0,l,0)$.

By Proposition A.3 in Appendix A (applied with $E(s)\equiv\{0\}\oplus\{0\}\oplus\mathbb{R}^m$), it follows that the spectral flow of $A(s)$ coincides with the Robbin-Salamon index of the degenerate path $s\mapsto M(s,1)$. Thus

$$\operatorname{ind}D_{(u,\lambda)}=\mu_{RS}\left(M(\Psi(s,1),X(s,1),E(s,1)),s\in\mathbb{R}\right).$$

By the *(Homotopy)* and *(Catenation)* axioms for the Robbin-Salamon index [20], and using that $\lim_{s\to-\infty}M(s,\theta)=\overline{M}(\theta)$ and $\lim_{s\to\infty}M(s,\theta)=\underline{M}(\theta)$, we obtain

$$\begin{aligned}\operatorname{ind}D_{(u,\lambda)} &= \mu_{RS}\left(M(\underline{\Psi}(\theta),\underline{X}(\theta),\underline{E}(\theta)),\theta\in[0,1]\right) \\ &\quad -\mu_{RS}\big(M(\overline{\Psi}(\theta),\overline{X}(\theta),\overline{E}(\theta)),\theta\in[0,1]\big) \\ &= \mu(\underline{\gamma},\underline{\lambda})-\mu(\overline{\gamma},\overline{\lambda}).\end{aligned}$$

□

3.3 The parametrized chain complex

Given $H\in\mathcal{H}_{\Lambda,\mathrm{reg}}$, $(J,g)\in\mathcal{J}_{\mathrm{reg}}(H)$, and a free homotopy class a in $\widehat{W}$, we define $SC_*^{a,\Lambda}(H,J,g)$ as a chain complex whose underlying Λ_ω-module is

$$SC_*^{a,\Lambda}(H,J,g):=\bigoplus_{p\in\mathcal{P}^a(H)}\Lambda_\omega\langle p\rangle.$$

We define the degree of a generator $p \in \mathcal{P}(H)$ in terms of the parametrized Robbin-Salamon index by

$$|p| := -\mu(p) + \frac{m}{2} \in \mathbb{Z}.$$

The fact that the grading is integral follows from [20, Theorem 4.7] (we recall that $m = \dim \Lambda$). We define $|p\, e^A| := |p| - 2\langle c_1(T\widehat{W}), A\rangle$, where $c_1(T\widehat{W})$ is computed with respect to a compatible almost complex structure.

Recall that, for each $p = (\gamma, \lambda) \in \mathcal{P}(H)$, we have chosen a cylinder $\sigma_p : [0,1] \times S^1 \to \widehat{W}$ such that $\sigma_p(0,\cdot) = l_{[\gamma]}$ and $\sigma_p(1,\cdot) = \gamma$. We define $\overline{\sigma}_p(s,\theta) := \sigma_p(1-s,\theta)$. Given $\overline{p} = (\overline{\gamma}, \overline{\lambda}), \underline{p} = (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ we define

$$\mathcal{M}^A(\overline{p}, \underline{p}; H, J, g) \subset \mathcal{M}(\overline{p}, \underline{p}; H, J, g)$$

to consist of trajectories (u, λ) such that $[\sigma_{\overline{p}} \# u \# \overline{\sigma}_{\underline{p}}] = A \in H_2(\widehat{W}; \mathbb{Z})$. It follows from Theorem 3.5 that

$$\dim \mathcal{M}^A(\overline{p}, \underline{p}; H, J, g) = |\overline{p}| - |\underline{p}\, e^A| - 1.$$

Let $\overline{p} := (\overline{\gamma}, \overline{\lambda}), \underline{p} := (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$. Whenever $|\overline{p}| - |\underline{p}\, e^A| = 1$, one can associate to each element $(u, \lambda) \in \mathcal{M}^A(\overline{p}, \underline{p}; H, J, g)$ a sign $\varepsilon(u, \lambda)$ via the coherent orientations recipe of Floer and Hofer [12]. As in their construction, since the asymptotics are fixed, the relevant spaces of Fredholm operators are contractible, and the corresponding determinant line bundles are trivial. Hence the moduli spaces of parametrized Floer trajectories are orientable. Since our moduli spaces are modeled on $\mathbb{R}$ as gradient trajectories, we can use the algorithm in [12] to construct a set of orientations which is coherent with respect to the gluing operation. More precisely, one chooses an element $p \in \mathcal{P}(H)$, and for each $p \neq \underline{p} \in \mathcal{P}(H)$ one chooses arbitrary orientations of the spaces of operators $\mathcal{O}(p, \underline{p})$ asymptotic to D_p at $-\infty$ and to $D_{\underline{p}}$ at $+\infty$. These determine orientations of $\mathcal{O}(\underline{p}, p)$ by requiring that the glued orientation on $\mathcal{O}(p, p)$ be the one determined by the canonical orientation of the constant operator D_p. We obtain orientations on $\mathcal{O}(\overline{p}, \underline{p})$ by requiring that the glued orientation with $\mathcal{O}(p, \overline{p})$ and $\mathcal{O}(\underline{p}, p)$ be the canonical one on $\mathcal{O}(p, p)$.

We define a differential ∂ on $SC_*^{a,\Lambda}(H, J, g)$ by

$$\partial \overline{p} := \sum_{|\overline{p}| - |\underline{p}\, e^A| = 1} \Big(\sum_{(u,\lambda) \in \mathcal{M}^A(\overline{p}, \underline{p}; H, J, g)} \varepsilon(u, \lambda) \Big)\, \underline{p}\, e^A.$$

This expression is well-defined by standard compactness arguments [14, 22]. More precisely, for each $A \in H_2(W;\mathbb{Z})$ satisfying $|\overline{p}| - |\underline{p}\, e^A| = 1$ the set $\mathcal{M}^A(\overline{p}, \underline{p}; H, J, g)$ is finite, and for each $c > 0$ the number of $A \in H_2(W;\mathbb{Z})$ such that $\omega(A) \le c$ and $\mathcal{M}^A(\overline{p}, \underline{p}; H, J, g) \neq \emptyset$ is finite.

It follows from standard compactness and gluing arguments [11, 22] that $\partial^2 = 0$. Compactness is established in three steps. Firstly, one obtains a uniform C^0-bound on the $\widehat{W}$-component of parametrized Floer trajectories using the maximum principle and the fact that $\partial^2 H/\partial s \partial t = 0$ outside a compact set [18, Lemma 1.5]. Secondly, one proves that the Λ-component converges by applying the Arzelá-Ascoli theorem. Thirdly, the $\widehat{W}$-component converges by Floer-Gromov compactness because it satisfies an s-dependent Floer equation. Gluing involves exactly the same kind of estimates as in Floer theory.

We denote the resulting homology groups by $SH_*^{a,\Lambda}(H, J, g)$. As for usual symplectic homology, we obtain by passing to the direct limit **parametrized symplectic homology groups**

$$SH_*^{a,\Lambda}(W) := \lim_{\substack{\longrightarrow \\ H \in \mathcal{H}_{\Lambda,\mathrm{reg}}}} SH_*^{a,\Lambda}(H, J, g).$$

Proposition 3.7 (Künneth formula)**.** *The following isomorphism holds with field coefficients*

$$SH_*^{a,\Lambda}(W) \simeq SH_*^a(W) \otimes H_*(\Lambda). \tag{3.30}$$

Proof. We use Hamiltonians of the form

$$H_\lambda(\theta, x) := K(\theta, x) + f(\lambda).$$

Here $f : \Lambda \to \mathbb{R}$ is a Morse function and K is an admissible Hamiltonian having nondegenerate orbits. We choose a generic admissible almost complex structure J on W and a generic Riemannian metric g on Λ.

The critical points of the parametrized action functional are of the form (γ, λ), $\gamma \in \mathcal{P}(K)$, $\lambda \in \mathrm{Crit}(f)$. The properties of the parametrized Robbin-Salamon index described in Appendix A imply that

$$\mu(\gamma, \lambda) = \mu_{RS}(\gamma) + \mathrm{ind}_f(\lambda) - \frac{m}{2},$$

where $\mathrm{ind}_f(\lambda)$ denotes the Morse index of $\lambda \in \mathrm{Crit}(f)$. It follows that

$$|(\gamma, \lambda)| = -\mu_{RS}(\gamma) + m - \mathrm{ind}_f(\lambda) = -\mu_{RS}(\gamma) + \mathrm{ind}_{-f}(\lambda).$$

The parametrized Floer equation is split and has the form

$$\left\{ \begin{array}{rcl} \bar{\partial}_J u & = & JX_{H_\lambda} = JX_K, \\ \dot{\lambda}(s) & = & \int_{S^1} \vec{\nabla}_\lambda H(\theta, u(s,\theta), \lambda(s)) d\theta = \vec{\nabla} f(\lambda(s)). \end{array} \right.$$

This follows from the obvious identities

$$X_{H_\lambda}(\theta, x, \lambda) \equiv X_K(\theta, x), \qquad \vec{\nabla}_\lambda H(\theta, x, \lambda) \equiv \vec{\nabla} f(\lambda).$$

We obtain an isomorphism of complexes

$$SC_*^{a,\Lambda}(H, J, g) \simeq SC_*^a(K, J) \otimes C_*(-f, g),$$

where $SC_*^a(K, J)$ denotes the Floer complex for (K, J) in the free homotopy class a (graded by $-\mu_{RS}(\gamma)$) and $C_*(-f, g)$ denotes the Morse complex for $(-f, g)$ (graded by $\mathrm{ind}_{-f}(\lambda)$). Since we use field coefficients the conclusion follows by the algebraic Künneth theorem. $\square$

Remark 3.8. (Naturality) An embedding of parameter spaces $\iota : \Lambda \hookrightarrow \Lambda'$ induces a natural map $S\iota_* : SH_*^{a,\Lambda}(W) \to SH_*^{a,\Lambda'}(W)$ which is equal to $\mathrm{Id} \otimes \iota_*$ via the Künneth isomorphism. This can be seen by using a Hamiltonian $K(\theta, x) + f(\lambda)$ on $S^1 \times \widehat{W} \times \Lambda$ as in the proof of Proposition 3.7 above, and a Hamiltonian $K(\theta, x) + \widetilde{f}(\lambda')$ on $S^1 \times \widehat{W} \times \Lambda'$, where $\widetilde{f} = f + |y|^2$ in a tubular neighbourhood of $\Lambda \subset \Lambda'$ and y is the normal coordinate.

4 S^1-equivariant theories

In §4.1 we give a Morse theoretic presentation of S^1-equivariant homology of a manifold carrying an S^1-action. This serves as a motivation for §4.2, where we give the definition of the S^1-equivariant symplectic homology groups $SH_*^{S^1}(W)$ following Viterbo [27, §5]. We adopt a slightly more general setting and define groups $SH_*^{a,S^1}(W)$ corresponding to nontrivial free homotopy classes of loops in W.

4.1 S^1-equivariant homology and Morse theory

In this section M denotes a finite-dimensional smooth manifold carrying a smooth action of S^1. Our aim is to give a description of

$$H_*^{S^1}(M) := H_*(M \times_{S^1} ES^1)$$

in terms of Morse homology. We recall that $ES^1 = \varinjlim_N S^{2N+1}$ and therefore $M \times_{S^1} ES^1 = \varinjlim_N M \times_{S^1} S^{2N+1}$. We denote

$$M_{S^1} := M \times_{S^1} ES^1, \quad M_{S^1}^{(N)} := M \times_{S^1} S^{2N+1}.$$

The first observation is that, given a positive integer k, the homology groups $H_*(M_{S^1}^{(N)})$ stabilize in degree $* \le k$ for N large enough. Indeed, the equivariant inclusion $S^{2N+1} \hookrightarrow S^{2N+3}$ induces an inclusion of fibrations

$$\begin{tikzcd} M \arrow[r, hook] & M_{S^1}^{(N)} \arrow[r] \arrow[d] & \mathbb{C}P^N \arrow[d] \\ M \arrow[r, hook] & M_{S^1}^{(N+1)} \arrow[r] & \mathbb{C}P^{N+1} \end{tikzcd}$$

This induces in turn a morphism between the associated Leray-Serre spectral sequences which is an isomorphism on the E^2-page in total degree less than N. Functoriality of the Leray-Serre spectral sequence implies that, for N large enough (and determined by k), the above inclusion induces isomorphisms $H_*(M_{S^1}^{(N)}) \xrightarrow{\sim} H_*(M_{S^1}^{(N+1)})$, $* \le k$ (see for example [16, Theorem 3.5]).

We can give a description of $H_*(M_{S^1}^{(N)})$ in terms of Morse-Bott functions on $M \times S^{2N+1}$ as follows. We choose a function $a : M \times S^{2N+1} \to \mathbb{R}$ which is S^1-invariant, i.e.

$$a(\tau x, \tau\lambda) = a(x, \lambda), \quad \tau \in S^1,\ (x, \lambda) \in M \times S^{2N+1},$$

and which has only Morse-Bott circles of critical points, i.e. the induced function $\underline{a} : M_{S^1}^{(N)} \to \mathbb{R}$ is Morse. We denote by S_p, $p \in \mathrm{Crit}(a)$ these circles of critical points and by $[p] \in M_{S^1}^{(N)}$ the nondegenerate critical point of $\underline{a}$ corresponding to S_p, so that $S_p = S_{\tau \cdot p}$ and $[p] = [\tau \cdot p]$, $\tau \in S^1$. We denote by

$$\mathrm{ind}(S_p) = \mathrm{ind}([p])$$

the Morse-Bott index of S_p.

We choose a generic S^1-invariant metric g on $M \times S^{2N+1}$ such that the gradient flow of a has the Thom-Smale transversality property, i.e.

$$W^u(S_p) \pitchfork W^s(S_q), \quad p, q \in \mathrm{Crit}(a).$$

This is equivalent to asking that the gradient flow of $\underline{a}$ with respect to the induced metric $\underline{g}$ on $M^{(N)}_{S^1}$ satisfies $W^u([p]) \pitchfork W^s([q])$, $[p],[q] \in \mathrm{Crit}(\underline{a})$. Given $\overline{p}, \underline{p} \in \mathrm{Crit}(a)$ we denote by

$$\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; a, g)$$

the **space of gradient trajectories** consisting of maps $v = (u, \lambda) : \mathbb{R} \to M \times S^{2N+1}$ which satisfy

$$\dot{v} = -\vec{\nabla} a(v) \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \dot{u} & = & -\vec{\nabla}_x a(u, \lambda), \\ \dot{\lambda} & = & -\vec{\nabla}_\lambda a(u, \lambda), \end{array} \right. \tag{4.1}$$

and

$$\left\{ \begin{array}{rcl} \lim\limits_{s \to -\infty} v(s) & \in & S_{\overline{p}}, \\ \lim\limits_{s \to \infty} v(s) & \in & S_{\underline{p}}, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \lim\limits_{s \to -\infty} (u(s), \lambda(s)) & = & (\overline{x}, \overline{\lambda}) \in S_{\overline{p}}, \\ \lim\limits_{s \to \infty} (u(s), \lambda(s)) & = & (\underline{x}, \underline{\lambda}) \in S_{\underline{p}}. \end{array} \right. \tag{4.2}$$

Here the gradient $\vec{\nabla}$ is considered with respect to the metric g and $\vec{\nabla}_x, \vec{\nabla}_\lambda$ are its components along TM and TS^{2N+1} respectively. Under the transversality assumption for the metric g the space of gradient trajectories is a smooth manifold of dimension

$$\dim \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; a, g) = \mathrm{ind}(S_{\overline{p}}) - \mathrm{ind}(S_{\underline{p}}) + 1.$$

It carries a natural action of $\mathbb{R}$ by reparametrization and we denote by

$$\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; a, g) := \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; a, g)/\mathbb{R}$$

the **moduli space of gradient trajectories**. In our setting the moduli space carries an action of S^1 and the quotient

$$\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; a, g) := \mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; a, g)/S^1$$

is a smooth manifold of dimension

$$\dim \mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; a, g) = \mathrm{ind}(S_{\overline{p}}) - \mathrm{ind}(S_{\underline{p}}) - 1.$$

The bundle with fiber $TW^u(\tau \cdot p)$, $\tau \in S^1$ over S_p is orientable since $W^u(S_p) := \bigcup_{\tau \in S^1} W^u(\tau \cdot p)$ carries an action of S^1. We choose for each S_p an orientation of this bundle, which amounts to choosing an orientation of

$W^u(S_p)$. Since each S_p inherits a natural orientation from S^1, this determines a coorientation of the bundle with fiber $TW^s(\tau \cdot p)$, $\tau \in S^1$ over S_p and therefore a coorientation of $W^s(S_p) := \bigcup_{\tau \in S^1} W^s(\tau \cdot p)$. We get orientations on $\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; a, g)$ and, after quotienting out $\mathbb{R}$ and S^1, we get orientations on $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; a, g)$, $\overline{p}, \underline{p} \in \mathrm{Crit}(a)$. In particular, if $\mathrm{ind}(\overline{p}) - \mathrm{ind}(\underline{p}) = 1$ the moduli space $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; a, g)$ is zero-dimensional and each element $[v]$ inherits a sign $\epsilon([v])$.

We define the **S^1-equivariant Morse complex** by

$$C_k^{S^1}(a, g) := \bigoplus_{\mathrm{ind}(S_p)=k} \mathbb{Z}\langle S_p \rangle,$$

with the **S^1-equivariant Morse differential**

$$d^{S^1} : C_k^{S^1} \to C_{k-1}^{S^1},$$

$$d^{S^1}\langle S_{\overline{p}} \rangle := \sum_{\mathrm{ind}(S_{\overline{p}}) - \mathrm{ind}(S_{\underline{p}}) = 1} \; \sum_{[v] \in \mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; a, g)} \epsilon([v]) \langle S_{\underline{p}} \rangle.$$

Since the elements of $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; a, g)$ are in one-to-one correspondence with elements of the moduli space $\mathcal{M}([\overline{p}], [\underline{p}]; \underline{a}, \underline{g})$ of gradient trajectories of $\underline{a}$ with respect to the metric $\underline{g}$ on $M_{S^1}^{(N)}$, and since the rule for obtaining signs on $\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; a, g)$ if $\mathrm{ind}(S_{\overline{p}}) - \mathrm{ind}(S_{\underline{p}}) = 1$ induces the usual Morse homology rule for signs on $\mathcal{M}([\overline{p}], [\underline{p}]; \underline{a}, \underline{g})$, we infer that the complex $(C_*^{S^1}, d^{S^1})$ is tautologically isomorphic with the Morse complex of the pair $(\underline{a}, \underline{g})$. Therefore

$$H_k(C_*^{S^1}, d^{S^1}) \simeq H_k(M_{S^1}^{(N)}), \quad k \in \mathbb{N}$$

and, for N large enough (depending on k), we have

$$H_k(C_*^{S^1}, d^{S^1}) \simeq H_k^{S^1}(M).$$

Remark 4.1. The previous construction admits an obvious reformulation for any manifold P endowed with a free S^1-action: the homology of the quotient P/S^1 can be described in terms of Morse-Bott data on P alone.

4.2 S^1-equivariant symplectic homology

In this section we give the definition of S^1-equivariant symplectic homology following Viterbo [27]. Our treatment parallels the finite dimensional case as presented in §4.1. We obtain the definition of S^1-equivariant symplectic homology as a variant of parametrized symplectic homology with $\Lambda = S^{2N+1}$.

The space of smooth loops $\gamma : S^1 \to \widehat{W}$ carries an action of S^1 given by

$$(\tau \cdot \gamma)(\cdot) := \gamma(\cdot - \tau), \quad \tau \in S^1.$$

Let $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ be a family of Hamiltonian functions denoted by $H(\theta, x, \lambda) = H_\lambda(\theta, x)$. This defines a family of action functionals

$$\mathcal{A} : C^\infty(S^1, \widehat{W}) \times S^{2N+1} \to \mathbb{R},$$

$$\mathcal{A}(\gamma, \lambda) = \mathcal{A}_\lambda(\gamma) := -\int_{[0,1]\times S^1} \sigma^*\omega - \int_{S^1} H_\lambda(\theta, \gamma(\theta))d\theta,$$

where $\sigma : [0,1] \times S^1 \to \widehat{W}$ is a smooth homotopy from $l_{[\gamma]}$ to γ, and $l_{[\gamma]}$ is a fixed representative of the free homotopy class of γ.

Lemma 4.2. *The family $\mathcal{A}$ is invariant with respect to the diagonal action of S^1 if and only if the family of Hamiltonians satisfies*

$$H_{\tau\lambda}(\theta + \tau, \cdot) = H_\lambda(\theta, \cdot) + r(\theta, \tau, \lambda) \tag{4.3}$$

for some function $r : S^1 \times S^1 \times S^{2N+1} \to \mathbb{R}$ such that

$$\int_{S^1} r(\theta, \tau, \lambda)d\theta = 0 \text{ for all } \tau \in S^1,\ \lambda \in S^{2N+1} \tag{4.4}$$

and

$$r(\theta, 1, \lambda) = 0, \qquad r(\theta + \tau, -\tau, \tau\lambda) = -r(\theta, \tau, \lambda). \tag{4.5}$$

Proof. The nontrivial implication is that invariance of $\mathcal{A}$ implies the desired condition on H. We thus assume that $\mathcal{A}$ is invariant, i.e. $\mathcal{A}_{\tau\lambda}(\tau\gamma) = \mathcal{A}_\lambda(\gamma)$ for all loops γ. This is equivalent to the equality

$$\int_{S^1} H_{\tau\lambda}(\theta + \tau, \gamma(\theta))d\theta = \int_{S^1} H_\lambda(\theta, \gamma(\theta))d\theta,\ \forall\ \gamma$$

and, denoting $F(\theta,\tau,\lambda,x) := H_{\tau\lambda}(\theta+\tau,x) - H_\lambda(\theta,x)$, we obtain

$$\int_{S^1} F(\theta,\tau,\lambda,\gamma(\theta))d\theta = 0,\ \forall\ \gamma,\tau,\lambda.$$

By letting γ vary in the neighbourhood of the constant loop at some $x \in \widehat{W}$ we see that we must have $\int_{S^1} D_x F(\theta,\tau,\lambda,x)\cdot\zeta(\theta)d\theta = 0$ for all loops ζ of tangent vectors at x. It follows that $D_x F(\theta,\tau,\lambda,x) = 0$ for all $\theta \in S^1$ and, since x was chosen arbitrarily, we get $F(\theta,\tau,\lambda,x) = r(\theta,\tau,\lambda)$ with $\int_{S^1} r(\theta,\tau,\lambda)d\theta = 0$. This shows (4.4), whereas (4.5) is straightforward. $\square$

Remark 4.3. Condition (4.3) holds for example if $r \equiv 0$, i.e. if the family H satisfies

$$H_{\tau\lambda}(\theta+\tau,\cdot) = H_\lambda(\theta,\cdot). \tag{4.6}$$

In particular one can choose the family H to be given by a single autonomous Hamiltonian $H(\theta,x,\lambda) = H(x)$.

We denote by $\mathcal{H}_N^{S^1} \subset \mathcal{H}_{S^{2N+1}}$ the set of admissible Hamiltonian families $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ satisfying condition (4.6). It follows from the definitions that there exists $t_0 \geq 0$ such that, for $t \geq t_0$, we have $H(\theta,p,t,\lambda) = \beta e^t + \beta'(\lambda)$, with $0 < \beta \notin \mathrm{Spec}(M,\alpha)$, and $\beta' \in C^\infty(S^{2N+1},\mathbb{R})$ invariant under the action of S^1.

The differential of $\mathcal{A}$ is given by (3.1) and critical points of $\mathcal{A}$ satisfy (3.2). Since $\mathcal{A}$ is S^1-invariant, the set of critical points of $\mathcal{A}$ is S^1-invariant as well, i.e. if $(\gamma,\lambda) \in \mathcal{P}(H)$, then $(\tau\gamma,\tau\lambda) \in \mathcal{P}(H)$ for all $\tau \in S^1$. Given $p := (\gamma,\lambda) \in \mathcal{P}(H)$ we denote

$$S_p = S_{(\gamma,\lambda)} := \{(\tau\gamma,\tau\lambda)\ :\ \tau \in S^1\} \subset \mathcal{P}(H),$$

so that $S_p = S_{\tau\cdot p}$, $\tau \in S^1$. We shall refer to S_p as an **S^1-orbit of critical points** (of $\mathcal{A}$).

An **admissible family of almost complex structures** $J = (J_\lambda^\theta)$ (in the sense of Section 3) is called **S^1-invariant** if it satisfies the condition

$$J_{\tau\lambda}^{\theta+\tau} = J_\lambda^\theta, \qquad \theta \in S^1,\ \tau \in S^1,\ \lambda \in S^{2N+1}. \tag{4.7}$$

Such a J^θ induces an S^{2N+1}-family of L^2-metrics on $C^\infty(S^1,\widehat{W})$ defined by

$$\langle\zeta,\eta\rangle_\lambda := \int_{S^1} \omega(\zeta(\theta), J_\lambda^\theta\eta(\theta))d\theta, \quad \zeta,\eta \in T_\gamma C^\infty(S^1,\widehat{W}) = \Gamma(\gamma^*T\widehat{W}).$$

Condition (4.7) ensures that, when coupled with an S^1-invariant metric g on S^{2N+1}, this family gives rise to an S^1-invariant metric on $C^\infty(S^1, \widehat{W}) \times S^{2N+1}$. We denote by $\mathcal{J}_N^{S^1}$ the set of pairs (J, g) consisting of an S^1-invariant admissible family of almost complex structures J on $\widehat{W}$ and of an S^1-invariant Riemannian metric g on S^{2N+1}.

Given $H \in \mathcal{H}_N^{S^1}$, $(J, g) \in \mathcal{J}_N^{S^1}$, and $\overline{p} := (\overline{\gamma}, \overline{\lambda}), \underline{p} := (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$, we denote by

$$\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$$

the **space of S^1-equivariant Floer trajectories**, consisting of pairs (u, λ) with

$$u : \mathbb{R} \times S^1 \to \widehat{W}, \qquad \lambda : \mathbb{R} \to S^{2N+1},$$

satisfying

$$\begin{aligned} \partial_s u + J^\theta_{\lambda(s)} \partial_\theta u - J^\theta_{\lambda(s)} X^\theta_{H_{\lambda(s)}}(u) &= 0, && (4.8) \\ \dot{\lambda}(s) - \int_{S^1} \vec{\nabla}_\lambda H(\theta, u(s,\theta), \lambda(s)) d\theta &= 0, && (4.9) \end{aligned}$$

and

$$\lim_{s \to -\infty} (u(s, \cdot), \lambda(s)) \in S_{\overline{p}}, \quad \lim_{s \to +\infty} (u(s, \cdot), \lambda(s)) \in S_{\underline{p}}. \qquad (4.10)$$

The additive group $\mathbb{R}$ acts on $\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$ by reparametrization in the s-variable. We denote by

$$\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, J, g) := \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)/\mathbb{R}$$

the **moduli space of S^1-equivariant Floer trajectories**. This space is endowed with natural evaluation maps

$$\overline{\mathrm{ev}} : \mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, J, g) \to S_{\overline{p}}, \qquad \underline{\mathrm{ev}} : \mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, J, g) \to S_{\underline{p}}.$$

An S^1-orbit of critical points $S_p \subset \mathcal{P}(H)$ is called **nondegenerate** if the Hessian $d^2\mathcal{A}(\gamma, \lambda)$ has a 1-dimensional kernel V_p for some (and hence any) $(\gamma, \lambda) \in S_p$. It follows from [5, Lemma 2.3] that nondegeneracy is equivalent to the fact that the kernel of the asymptotic operator D_p is also 1-dimensional and equal to V_p. In both cases, a generator of V_p is given by the infinitesimal generator of the S^1-action.

We define the set $\mathcal{H}_{N,\mathrm{reg}}^{S^1} \subset \mathcal{H}_N^{S^1}$ to consist of elements H such that, for any $p \in \mathcal{P}(H)$, the S^1-orbit S_p is nondegenerate. We proved in [5, Proposition 5.1]

that the set $\mathcal{H}^{S^1}_{N,\mathrm{reg}}$ is of the second Baire category in $\mathcal{H}^{S^1}_N$. Moreover, if $H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}$, each S^1-orbit $S_p \subset C^\infty(S^1, \widehat{W}) \times S^{2N+1}$ is isolated.

Let $d > 0$ be small enough (for a fixed $H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}$, one can take $d > 0$ to be smaller than the minimal spectral gap of the asymptotic operators D_p, $p \in \mathcal{P}(H)$), and fix $1 < p < \infty$. Given $\overline{p}, \underline{p} \in \mathcal{P}(H)$ and $(u,\lambda) \in \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$, we define

$$\begin{aligned}\mathcal{W}^{1,p,d} &:= W^{1,p}(u^*T\widehat{W}; e^{d|s|}dsd\theta) \oplus W^{1,p}(\lambda^*TS^{2N+1}; e^{d|s|}ds) \oplus V_{\overline{p}} \oplus V_{\underline{p}},\\ \mathcal{L}^{p,d} &:= L^p(u^*T\widehat{W}; e^{d|s|}dsd\theta) \oplus L^p(\lambda^*TS^{2N+1}; e^{d|s|}ds).\end{aligned}$$

Here we identify $V_{\overline{p}}$, $V_{\underline{p}}$ with the 1-dimensional spaces generated by the sections $\beta(s)(\dot{\overline{\gamma}}, X_{\overline{\lambda}})$, respectively $\beta(-s)(\dot{\underline{\gamma}}, X_{\underline{\lambda}})$ of $u^*T\widehat{W} \oplus \lambda^*TS^{2N+1}$. For this identification, we denote by $X_{\overline{\lambda}}$, $X_{\underline{\lambda}}$ the values of the infinitesimal generator of the S^1-action on S^{2N+1} at the points $\overline{\lambda}$, respectively $\underline{\lambda}$, and choose a cut-off function $\beta : \mathbb{R} \to [0,1]$ which is equal to 1 near $-\infty$, and vanishes near $+\infty$.

Proposition 4.4. *Assume $S_{\overline{p}}, S_{\underline{p}} \subset \mathcal{P}(H)$ are nondegenerate. For any $(u,\lambda) \in \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$ the operator*

$$D_{(u,\lambda)} : \mathcal{W}^{1,p,d} \to \mathcal{L}^{p,d}$$

is Fredholm of index

$$\mathrm{ind}\, D_{(u,\lambda)} = -\mu(\overline{p}) + \mu(\underline{p}) + 1.$$

In the above statement, it is understood that the trivialization used to define $\mu(\overline{p})$ is obtained from the trivialization used to define $\mu(\underline{p})$ by continuation along the map u.

Proof. The Fredholm property was proved in [5, Proposition 5.2] as follows. Let $\mathcal{W}^{1,p}$ and $\mathcal{L}^p$ be defined as $\mathcal{W}^{1,p,d}$ and $\mathcal{L}^{p,d}$ above, with $d = 0$ and without taking into account the direct summands $V_{\overline{p}}$, $V_{\underline{p}}$. Let $\widetilde{D}_{(u,\lambda)} : \mathcal{W}^{1,p} \to \mathcal{L}^p$ be the operator obtained by conjugating with $e^{\frac{d}{p}|s|}$ the restriction of $D_{(u,\lambda)}$ to $W^{1,p}(u^*T\widehat{W}; e^{d|s|}dsd\theta) \oplus W^{1,p}(\lambda^*TS^{2N+1}; e^{d|s|}ds)$. Then $\widetilde{D}_{(u,\lambda)}$ has nondegenerate asymptotics, hence it is Fredholm by [5, Theorem 2.6]. Since the restriction of $D_{(u,\lambda)}$ to a codimension 2 subspace is conjugate to $\widetilde{D}_{(u,\lambda)}$, it follows that $D_{(u,\lambda)}$ is Fredholm as well.

The asymptotic operator at $-\infty$ is $\widetilde{D}_{\overline{p}} = D_{\overline{p}} + \frac{d}{p}\mathbb{1}$, and the asymptotic operator at $+\infty$ is $\widetilde{D}_{\underline{p}} = D_{\underline{p}} - \frac{d}{p}\mathbb{1}$. The parametrized Robbin-Salamon indices after perturbation are given by $\mu(\overline{p}) + \frac{1}{2}$, respectively $\mu(\underline{p}) - \frac{1}{2}$ (this can be seen for example using a Taylor expansion at order 1 in ε for a perturbation of the asymptotic operators of the form $\widetilde{D}_p = D_p + \varepsilon\mathbb{1}$). Using Theorem 3.5 we obtain

$$\begin{aligned}
\operatorname{ind} D_{(u,\lambda)} &= \operatorname{ind} \widetilde{D}_{(u,\lambda)} + 2 \\
&= -(\mu(\overline{p}) + \frac{1}{2}) + (\mu(\underline{p}) - \frac{1}{2}) + 2 \\
&= -\mu(\overline{p}) + \mu(\underline{p}) + 1.
\end{aligned}$$

□

Let $H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}$. A pair $(J,g) \in \mathcal{J}^{S^1}_N$ is called **regular for** $\boldsymbol{H}$ if the operator $D_{(u,\lambda)}$ is surjective for any $\overline{p}, \underline{p} \in \mathcal{P}(H)$ and any $(u,\lambda) \in \widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)$. We denote the set of such regular pairs by $\mathcal{J}^{S^1}_{N,\mathrm{reg}}(H)$.

We defined in [5, §7] two special classes $\mathcal{H}_*\mathcal{J}' \subset \mathcal{H}\mathcal{J}'$ in $\mathcal{H}^{S^1}_N \times \mathcal{J}^{S^1}_N$. We proved in [5, Theorem 7.4] that there exists an open subset $\mathcal{H}\mathcal{J}'_{\mathrm{reg}} \subset \mathcal{H}\mathcal{J}'$ which is dense in a neighbourhood of $\mathcal{H}_*\mathcal{J}' \subset \mathcal{H}\mathcal{J}'$, and which consists of triples (H,J,g) such that

$$H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}, \qquad (J,g) \in \mathcal{J}^{S^1}_{N,\mathrm{reg}}(H).$$

Let $(H,J,g) \in \mathcal{H}\mathcal{J}'_{\mathrm{reg}}$. Recall that, for each $p = (\gamma,\lambda) \in \mathcal{P}(H)$, we have chosen a cylinder $\sigma_p : [0,1]\times S^1 \to \widehat{W}$ such that $\sigma_p(0,\cdot) = l_{[\gamma]}$ and $\sigma_p(1,\cdot) = \gamma$. We define $\overline{\sigma}_p(s,\theta) := \sigma_p(1-s,\theta)$. Given $\overline{p} = (\overline{\gamma}, \overline{\lambda}), \underline{p} = (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ we define

$$\mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H, J, g) \subset \mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$$

to consist of trajectories (u,λ) such that $[\sigma_{\overline{p}} \# u \# \overline{\sigma}_{\underline{p}}] = A \in H_2(\widehat{W};\mathbb{Z})$. It follows from Proposition 4.4 that

$$\dim \mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H, J, g) = -\mu(\overline{p}) + \mu(\underline{p}) + 2\langle c_1(T\widehat{W}), A\rangle. \tag{4.11}$$

Since $\mathcal{A}$ and (J,g) are S^1-invariant, the moduli space $\mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$ carries a free action of S^1 induced by the diagonal action on $C^\infty(S^1, \widehat{W}) \times S^{2N+1}$, i.e.

$$\tau \cdot (u,\lambda) := (u(\cdot, \cdot - \tau), \tau\lambda).$$

We denote the quotient by

$$\mathcal{M}_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, J, g) := \mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)/S^1.$$

This is a smooth manifold of dimension

$$\dim \mathcal{M}^A_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, J, g) = -\mu(\overline{p}) + \mu(\underline{p}) + 2\langle c_1(T\widehat{W}), A\rangle - 1.$$

An important feature of these moduli spaces is that they admit a system of coherent orientations in the sense of [12]. The difference with respect to the setup of Floer homology is that the asymptotes for the moduli spaces are not fixed, but can vary along circles S_p, $p = (\gamma, \lambda) \in \mathcal{P}(H)$. However, if one chooses the trivializations of $\gamma^* T\widehat{W} \oplus T_\lambda S^{2N+1}$ so that they are invariant under the S^1-action, then the analytical expression of the asymptotic operators D_p, $p \in \mathcal{P}(H)$ only depends on S_p. It then follows from the arguments in [12] that the spaces of Fredholm operators of the form (3.11) with nondegenerate asymptotics of the form D_p, $p \in \mathcal{P}(H)$ are contractible, and hence the corresponding determinant line bundles are orientable. The system of coherent orientations on the moduli spaces $\mathcal{M}^A_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$ is obtained by pulling back a system of coherent orientations on these spaces of Fredholm operators, as in [12].

Given a free homotopy class a in $\widehat{W}$, we define the $\boldsymbol{S^1}$**-equivariant chain complex** $SC^{a,S^1,N}_*(H, J, g)$ as a chain complex whose underlying Λ_ω-module is

$$SC^{a,S^1,N}_*(H) := SC^{a,S^1,N}_*(H, J, g) := \bigoplus_{S_p \subset \mathcal{P}^a(H)} \Lambda_\omega\langle S_p\rangle. \tag{4.12}$$

The grading is defined by $|S_p\, e^A| := -\mu(p) + \frac{m}{2} - 2\langle c_1(T\widehat{W}), A\rangle$. The $\boldsymbol{S^1}$**-equivariant differential** $\partial^{S^1} : SC^{a,S^1,N}_*(H) \to SC^{a,S^1,N}_{*-1}(H)$ is defined by

$$\partial^{S^1}(S_{\overline{p}}) := \sum_{\substack{S_{\underline{p}} \subset \mathcal{P}^a(H) \\ |S_{\overline{p}}| - |S_{\underline{p}}\, e^A| = 1}} \; \sum_{[u] \in \mathcal{M}^A_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)} \epsilon([u]) S_{\underline{p}}\, e^A.$$

The sign $\epsilon([u])$ is obtained by comparing the coherent orientation of the moduli space $\mathcal{M}^A_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$ with the orientation induced by the infinitesimal generator of the S^1-action.

Proposition 4.5. *The map ∂^{S^1} satisfies*

$$\partial^{S^1} \circ \partial^{S^1} = 0.$$

The proof of Proposition 4.5 is given in Section 5.1. We define the $\boldsymbol{S^1}$**-equivariant Floer homology groups** by

$$SH_*^{a,S^1,N}(H,J,g) := H_*(SC_*^{a,S^1,N}(H), \partial^{S^1}).$$

Proposition 4.6. *Let $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}$. Given $(J_1,g_1),(J_2,g_2) \in \mathcal{J}_{N,\mathrm{reg}}^{S^1}(H)$, there exists a canonical isomorphism*

$$SH_*^{a,S^1,N}(H,J_1,g_1) \simeq SH_*^{a,S^1,N}(H,J_2,g_2).$$

We prove Proposition 4.6 in Section 5.2. Given $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}$ we shall denote $SH_*^{a,S^1,N}(H) := SH_*^{a,S^1,N}(H,J,g)$ for $(J,g) \in \mathcal{J}_{N,\mathrm{reg}}^{S^1}(H)$. In analogy with the construction of symplectic homology, we define

$$SH_*^{a,S^1,N}(W) := \lim_{\substack{\longrightarrow \\ H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}}} SH_*^{a,S^1,N}(H).$$

The $\boldsymbol{S^1}$**-equivariant symplectic homology groups of** $\boldsymbol{W}$ are defined by

$$SH_*^{a,S^1}(W) := \lim_{\substack{\longrightarrow \\ N}} SH_*^{a,S^1,N}(W).$$

The direct limit is taken with respect to the embeddings $S^{2N+1} \hookrightarrow S^{2N+3}$, inducing maps $SH_*^{a,S^1,N}(W) \to SH_*^{a,S^1,N+1}(W)$ (see Remark 3.8).

For the particular case of the trivial homotopy class $a = 0$, we denote the S^1-equivariant symplectic homology groups by $SH_*^{S_1}(W)$. Given $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}$ we define the **parametrized reduced action functional** $\mathcal{A}^0 : C^\infty_{\mathrm{contr}}(S^1,\widehat{W}) \times S^{2N+1} \to \mathbb{R}$ by

$$\mathcal{A}^0(\gamma,\lambda) := -\int_{D^2} \sigma^*\widehat{\omega} - \int_{S^1} H(\theta,\gamma(\theta),\lambda)\,d\theta.$$

Here $\sigma : D^2 \to \widehat{W}$ is a smooth extension of γ, and $\mathcal{A}^0$ is well-defined due to assumption (1.3).

Similarly to the case of symplectic homology, we define a special cofinal class of Hamiltonian families $\mathcal{H}_N'^{S^1} \subset \mathcal{H}_N^{S^1}$, consisting of elements $H = (H_\lambda) \in \mathcal{H}_N^{S^1}$ such that $H_\lambda \in \mathcal{H}'$ for all $\lambda \in S^{2N+1}$ (see Section 2 for the definition of the class $\mathcal{H}'$).

Given $H \in \mathcal{H}'^{S^1}_{N,\mathrm{reg}} := \mathcal{H}'^{S^1}_N \cap \mathcal{H}^{S^1}_{N,\mathrm{reg}}$, $(J,g) \in \mathcal{J}^{S^1}_{N,\mathrm{reg}}(H)$, and $\varepsilon > 0$ small enough, we define the chain complexes

$$SC^{-,S^1,N}_*(H,J,g) := \bigoplus_{\substack{S_p \subset \mathcal{P}^0(H) \\ \mathcal{A}^0(p) \le \varepsilon}} \Lambda_\omega \langle S_p \rangle \subset SC^{S^1,N}_*(H,J,g)$$

and

$$SC^{+,S^1,N}_*(H,J,g) := SC^{S^1,N}_*(H,J,g)/SC^{-,S^1,N}_*(H,J,g).$$

The differential on $SC^{\pm,S^1,N}_*(H,J,g)$ is induced by ∂^{S^1}. The corresponding homology groups $SH^{\pm,S^1,N}_*(H,J,g)$ do not depend on (J,g) and ε, and we define

$$SH^{\pm,S^1,N}_*(W) := \lim_{\substack{\longrightarrow \\ H \in \mathcal{H}'^{S^1}_{N,\mathrm{reg}}}} SH^{\pm,S^1,N}_*(H,J,g).$$

Passing to the direct limit over $N \to \infty$, we define

$$SH^{\pm,S^1}_*(W) := \lim_{\substack{\longrightarrow \\ N}} SH^{\pm,S^1,N}_*(W).$$

We call $SH^{+,S^1}_*(W)$ the **positive S^1-equivariant symplectic homology group** of (W,ω). It follows from the definitions that this fits into the **tautological long exact sequence**

$$\cdots \to SH^{+,S^1}_{k+1}(W) \to SH^{-,S^1}_k(W) \to SH^{S^1}_k(W) \to SH^{+,S^1}_k(W) \to \dots$$

Lemma 4.7. *Assume W has positive contact type boundary in the sense of Section 2. There is a natural isomorphism*

$$SH^{-,S^1}_*(W) \simeq H^{S^1}_{*+n}(W,\partial W;\Lambda_\omega).$$

Here $H^{S^1}_{*+n}(W,\partial W;\Lambda_\omega) \simeq H_{*+n}(W,\partial W;\Lambda_\omega) \otimes H_*(\mathbb{C}P^\infty;\mathbb{Q})$ denotes the S^1-equivariant homology of the pair $(W,\partial W)$ with respect to the trivial S^1-action.

Proof. We consider a Hamiltonian $H \in \mathcal{H}'^{S^1}_{N,\mathrm{reg}}$ which has the form

$$H(\theta,x,\lambda) = K(x) + \widetilde{f}(\lambda)$$

on $S^1 \times W \times S^{2N+1}$, with $K : W \to \mathbb{R}$ a C^2-small function, and $\widetilde{f} : S^{2N+1} \to \mathbb{R}$ the lift of a Morse function $f : \mathbb{C}P^N \to \mathbb{R}$. We choose $(J,g) \in \mathcal{J}^{S^1}_{N,\mathrm{reg}}(H)$ such that J is independent of θ and λ on W (this is possible because W

is symplectically aspherical [24]). The parametrized Floer equation is split, the Floer complex for (K, J) reduces to the Morse complex, and we have an isomorphism of complexes

$$SC_*^{-,S^1,N}(H, J, g) = C_{*+n}(K, J; \Lambda_\omega) \otimes C_*^{S^1}(\widetilde{f}, g; \mathbb{Q}).$$

Here C_* denotes the corresponding Morse complexes. Since $C_*^{S^1}(\widetilde{f}, g; \mathbb{Q})$ corresponds to a Morse complex on $\mathbb{C}P^N$, the conclusion follows. $\square$

5 Morse-Bott constructions

5.1 Morse-Bott complex for parametrized symplectic homology

We describe in this section a Morse-Bott construction for parametrized symplectic homology in the case when $\Lambda = S^{2N+1}$ and the action functional $\mathcal{A} : C^\infty(S^1, \widehat{W}) \times S^{2N+1} \to \mathbb{R}$ is S^1-invariant with respect to the diagonal action of S^1. The situation is analogous to that of Floer homology for an autonomous Hamiltonian considered in [3].

Let $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}$ and $(J, g) \in \mathcal{J}_{N,\mathrm{reg}}^{S^1}(H)$ as in Section 4.2. For each S^1-orbit of critical points $S_p \subset \mathcal{P}(H)$ we choose a perfect Morse function $f_p : S_p \to \mathbb{R}$. We denote by m_p, M_p the minimum, respectively the maximum of f_p. Given $\overline{p}, \underline{p} \in \mathcal{P}(H)$, $Q_{\overline{p}} \in \mathrm{Crit}(f_{\overline{p}})$, $Q_{\underline{p}} \in \mathrm{Crit}(f_{\underline{p}})$, and $m \geq 0$, we denote by

$$\mathcal{M}_m^A(Q_{\overline{p}}, Q_{\underline{p}}; H, \{f_p\}, J, g)$$

the union for $p_1, \ldots, p_{m-1} \in \mathcal{P}(H)$ and $A_1 + \cdots + A_m = A$ of the fibered products

$$W^u(Q_{\overline{p}}) \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A_1}(S_{\overline{p}}, S_{p_1}) \times \mathbb{R}^+)_{\varphi_{f_{p_1}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A_2}(S_{p_1}, S_{p_2}) \times \mathbb{R}^+)$$
$${}_{\varphi_{f_{p_2}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \cdots {}_{\varphi_{f_{p_{m-1}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \mathcal{M}^{A_m}(S_{p_{m-1}}, S_{\underline{p}})_{\underline{\mathrm{ev}}} \times W^s(Q_{\underline{p}}).$$

It follows from [3, Lemma 3.6] that, for a generic choice of the collection of Morse functions $\{f_p\}$, the previous fibered product is a smooth manifold of dimension

$$\begin{aligned} &\dim \mathcal{M}_m^A(Q_{\overline{p}}, Q_{\underline{p}}; H, \{f_p\}, J, g) \\ &\quad = \; -\mu(\overline{p}) + \mathrm{ind}_{f_{\overline{p}}}(Q_{\overline{p}}) + \mu(\underline{p}) - \mathrm{ind}_{f_{\underline{p}}}(Q_{\underline{p}}) + 2\langle c_1(T\widehat{W}), A \rangle - 1. \end{aligned}$$

We denote

$$\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H, \{f_p\}, J, g) := \bigcup_{m\geq 0} \mathcal{M}^A_m(Q_{\overline{p}}, Q_{\underline{p}}; H, \{f_p\}, J, g).$$

Given a free homotopy class a of loops in $\widehat{W}$, we define the **parametrized Morse-Bott chain complex** $BC_*^{a,N}(H, \{f_p\}, J, g)$ as a chain complex whose underlying Λ_ω-module is

$$BC_*^{a,N}(H) := BC_*^{a,N}(H, \{f_p\}, J, g) := \bigoplus_{S_p \subset \mathcal{P}^a(H)} \Lambda_\omega \langle m_p, M_p \rangle.$$

The grading is given by

$$\begin{aligned} |m_p\, e^A| &:= -\mu(\gamma, \lambda) + 1 - 2\langle c_1(T\widehat{W}), A\rangle, \\ |M_p\, e^A| &:= -\mu(\gamma, \lambda) - 2\langle c_1(T\widehat{W}), A\rangle. \end{aligned}$$

The **parametrized Morse-Bott differential**

$$d : BC_*^{a,N}(H) \to BC_{*-1}^{a,N}(H)$$

is defined by

$$dQ_{\overline{p}} := \sum_{\substack{\underline{p}\in\mathcal{P}^a(H), Q_{\underline{p}}\in \mathrm{Crit}(f_{\underline{p}}) \\ |Q_{\overline{p}}| - |Q_{\underline{p}}\, e^A| = 1}} \ \sum_{\mathbf{u}\in\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H, \{f_\gamma\}, J, g)} \epsilon(\mathbf{u}) Q_{\underline{p}}\, e^A, \qquad Q_{\overline{p}} \in \mathrm{Crit}(f_{\overline{p}}). \tag{5.1}$$

The sign $\varepsilon(\mathbf{u})$ is determined by the fibered-sum rule from coherent orientations on the relevant spaces of Fredholm operators, as explained in [3, Section 4.4].

The Correspondence Theorem 3.7 in [3] shows that $d^2 = 0$. Similarly to the construction of symplectic homology, we define

$$BH_*^{a,N}(W) := \varinjlim_{H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}} H_*(BC_*^{a,N}(H), d),$$

where the direct limit is taken with respect to increasing homotopies of Hamiltonians. It then follows from the Correspondence Theorem 3.7 in [3] that

$$BH_*^{a,N}(W) = SH_*^{a,S^{2N+1}}(W).$$

We now define a filtration on $BC_*^{a,N}(H)$ as follows. Let

$$B_k C_*^{a,N}(H) := \bigoplus_{\substack{S_p \subset \mathcal{P}^a(H) \\ A \in H_2(W;\mathbb{Z}) \\ -\mu(p) - 2\langle c_1(T\widehat{W}), A\rangle = k}} \langle m_p\, e^A,\ M_p\, e^A\rangle.$$

Proposition 5.1. *The $\mathbb{Q}$-vector spaces*

$$F_\ell BC_*^{a,S^1}(H) := \bigoplus_{k \le \ell} B_k C_*^{a,N}(H), \qquad \ell \in \mathbb{Z}$$

form an increasing filtration on $BC_*^{a,S^1}(H)$.

Proof. The formula (5.1) involves elements $Q_{\overline{p}}$, $Q_{\underline{p}}$ satisfying $|Q_{\overline{p}}| - |Q_{\underline{p}}\, e^A| = 1$, i.e.

$$-\mu(\overline{p}) + \mathrm{ind}_{f_{\overline{p}}}(Q_{\overline{p}}) + \mu(\underline{p}) - \mathrm{ind}_{f_{\underline{p}}}(Q_{\underline{p}}) + 2\langle c_1(T\widehat{W}), A\rangle = 1.$$

Since $\mathrm{ind}_{f_{\overline{p}}}(Q_{\overline{p}}) - \mathrm{ind}_{f_{\underline{p}}}(Q_{\underline{p}}) \in \{-1, 0, 1\}$, we obtain that $-\mu(\overline{p}) + \mu(\underline{p}) + 2\langle c_1(T\widehat{W}), A\rangle \in \{0, 1, 2\}$. $\square$

The differential d splits as

$$d = d^0 + d^1 + d^2$$

with $d^r : B_k C_*^{a,N}(H) \to B_{k-r} C_*^{a,N}(H)$. The complex $BC_*^{a,N}(H)$ admits a bi-grading which, for an element $Q_p\, e^A$ is $(-\mu(p) - 2\langle c_1(T\widehat{W}), A\rangle, \mathrm{ind}_{f_p}(Q_p))$. The associated spectral sequence $(E_{*,*}^{a,N;r}(H), \bar{d}^r)$ is supported in two lines and converges to $SH_*^{a,S^{2N+1}}(H)$. In particular, it degenerates at $r = 2$ and takes the form of a long exact sequence [4]

$$\cdots \longrightarrow SH_k^{a,S^{2N+1}}(H) \longrightarrow E_{k,0}^{a,N;2}(H) \xrightarrow{\bar{d}^2} E_{k-2,1}^{a,N;2}(H) \longrightarrow SH_{k-1}^{a,S^{2N+1}}(H) \longrightarrow \cdots \tag{5.2}$$

The differentials $\bar{d}^r$, $r = 0, 1, 2$ are induced by d^r. In particular, $\bar{d}^1$ satisfies

$$\bar{d}^1 \circ \bar{d}^1 = 0 \tag{5.3}$$

on $E_{*,*}^{a,N;1}(H)$.

Proposition 5.2. *For any $p \in \mathcal{P}^a(H)$ and any $B \in H_2(W;\mathbb{Z})$, the differential $d^0 : B_k C_*^{a,N}(H) \to B_k C_*^{a,N}(H)$ vanishes.*

Proof. Since d is Λ_ω-linear, it is enough to prove the statement for $B = 0$. By definition $d^0(Q_p)$, $Q_p \in \mathrm{Crit}(f_p)$ involves critical points of $f_{\underline{p}}$, $\underline{p} \in \mathcal{P}^a(H)$ satisfying $-\mu(p) + \mu(\underline{p}) + 2\langle c_1(T\widehat{W}), A\rangle = 0$. On the other hand, the dimension of the moduli spaces $\mathcal{M}^{A_1}(S_{p_1}, S_{p_2}; H, J, g)$ is equal to $-\mu(p_1) + \mu(p_2) + 2\langle c_1(T\widehat{W}), A_1\rangle$. Since these moduli spaces carry a free S^1-action, their dimension must be at least 1. This proves that $d^0(Q_p)$ counts only gradient trajectories of f_p emanating from Q_p. In particular $d^0(M_p) = 0$, and $d^0(m_p)$ is either 0 or equal to $\pm 2M_p$. That $d^0(m_p) = 0$ follows from the existence of a system of coherent orientations on the moduli spaces of S^1-equivariant Floer trajectories, by the same arguments as in [3, Lemma 4.28]. $\square$

As a consequence, provided that we work with rational coefficients, the term $E^{a,N;1}_{*,*}(H)$ can be expressed as

$$E^{a,N;1}_{*,*}(H) = \bigoplus_{S_p \subset \mathcal{P}^a(H)} \Lambda_\omega \langle m_p, M_p\rangle.$$

Let us denote by M the generator of $H_0(S^1)$ and by m the generator of $H_1(S^1)$. It follows from the definition (4.12) of the S^1-equivariant chain complex that there is a natural isomorphism of Λ_ω-modules which preserves the bi-degree

$$\Phi : E^{a,N;1}_{*,*}(H) \xrightarrow{\sim} SC^{a,S^1,N}_*(H) \otimes H_*(S^1),$$

given by

$$\Phi(m_p) := S_p \otimes m, \qquad \Phi(M_p) := S_p \otimes M.$$

Proposition 5.3. *There is a commutative diagram*

$$\begin{array}{ccc}
E^{a,N;1}_{*,*}(H) & \xrightarrow{\Phi} & SC^{a,S^1,N}_*(H) \otimes H_*(S^1) \\
\big\downarrow{\scriptstyle \bar{d}^1} & & \big\downarrow{\scriptstyle \partial^{S^1} \otimes \mathrm{Id}} \\
E^{a,N;1}_{*,*}(H) & \xrightarrow[\Phi]{} & SC^{a,S^1,N}_*(H) \otimes H_*(S^1)
\end{array}$$

Proof. By definition $\bar{d}^1(Q_{\overline{p}})$ involves critical points of $f_{\underline{p}}$, $\underline{p} \in \mathcal{P}^a(H)$ such that $-\mu(\overline{p}) + \mu(\underline{p}) + 2\langle c_1(T\widehat{W}), A\rangle = 1$. It follows from the dimension formula (4.11) that $\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H, \{f_p\}, J, g)$ involves exactly one parametrized Floer trajectory $u_1 \in \mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$. Since the dimension of the moduli space $\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H, \{f_p\}, J, g)$ is zero, it follows that either $\overline{\mathrm{ev}}(u_1) = M_{\overline{p}}$ and $Q_{\underline{p}} = M_{\underline{p}}$, or $\underline{\mathrm{ev}}(u_1) = m_{\underline{p}}$ and $Q_{\overline{p}} = m_{\overline{p}}$.

Using that the S^1-action on $\mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$ is free, we see that the coefficient of $Q_{\underline{p}}\, e^A$ in $\bar{d}^1(Q_{\overline{p}})$ is given by an algebraic count of connected components of $\mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$. The latter are in bijective correspondence with elements of $\mathcal{M}^A_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$, and the signs are the same by our convention for orienting the latter moduli space. Thus, the coefficient of $Q_{\underline{p}}\, e^A$ in $\bar{d}^1(Q_{\overline{p}})$ is equal to the coefficient of $S_{\underline{p}}\, e^A$ in $\partial^{S^1}(S_{\overline{p}})$. This proves the Proposition. □

Proof of Proposition 4.5. The claim $\partial^{S^1} \circ \partial^{S^1} = 0$ follows directly from Proposition 5.3, using that $\bar{d}^1 \circ \bar{d}^1 = 0$ (see (5.3)). □

It follows from Proposition 5.3 that Φ induces an isomorphism which respects the bi-degree

$$\bar{\Phi} : E^{a,N;2}_{*,*}(H) \xrightarrow{\sim} SH^{a,S^1,N}_*(H) \otimes H_*(S^1). \tag{5.4}$$

We are now ready to prove Theorem 1.1. We need two preparatory Lemmas.

Lemma 5.4. *We have* $\lim_{N\to\infty} SH^{S^{2N+1}}_*(W) = SH_*(W)$ *in each degree.*

Proof. The limit $\lim_{N\to\infty} SH^{S^{2N+1}}_*(W)$ is taken with respect to the maps $S\iota_*$ corresponding to the inclusions $\iota : S^{2N+1} \hookrightarrow S^{2N+3}$ as in Remark 3.8. Moreover, we saw that, modulo the Künneth isomorphism $SH^{S^{2N+1}}_*(W) \simeq SH_*(W) \otimes H_*(S^{2N+1})$ proved in Proposition 3.7, the map $S\iota_*$ is equal to $\mathrm{Id} \otimes \iota_*$. The conclusion follows. □

Lemma 5.5. *Let H_s be a smooth increasing homotopy from $H_0 \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}$ to $H_1 \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}$. Let $(J_i, g_i) \in \mathcal{J}^{S^1}_{N,\mathrm{reg}}(H_i)$, $i = 0, 1$ and (J_s, g_s) a regular smooth homotopy in $\mathcal{J}^{S^1}_N$ from (J_0, g_0) to (J_1, g_1). The induced chain morphism $BC^{a,S^1,N}_*(H_0, J_0, g_0) \to BC^{a,S^1,N}_*(H_1, J_1, g_1)$ respects the filtrations.*

The proof of Lemma 5.5 is given in Section 5.2 below.

Proof of Theorem 1.1. Using the isomorphism $\bar{\Phi}$ in (5.4), the long exact sequence (5.2) becomes

$$\ldots \to SH^{a,S^{2N+1}}_k(H) \to SH^{a,S^1,N}_k(H) \to SH^{a,S^1,N}_{k-2}(H) \to SH^{a,S^{2N+1}}_{k-1}(H) \to \ldots$$

By Lemma 5.5, a smooth increasing homotopy of Hamiltonian families in $\mathcal{H}^{S^1}_{N,\mathrm{reg}}$ induces a filtered chain morphism, and therefore a commutative diagram of exact sequences. Passing to the direct limit over $H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}}$ and using that the direct limit functor is exact, we obtain a long exact sequence

$$... \to SH_k^{a,S^{2N+1}}(W) \to SH_k^{a,S^1,N}(W) \to SH_{k-2}^{a,S^1,N}(W) \to SH_{k-1}^{a,S^{2N+1}}(W) \to ...$$

Passing further to the direct limit over $N \to \infty$, and using Lemma 5.4, we obtain

$$... \to SH_k^a(W) \to SH_k^{a,S^1}(W) \to SH_{k-2}^{a,S^1}(W) \to SH_{k-1}^a(W) \to ...$$

□

We will now prove Theorem 1.2. We need a preliminary algebraic lemma involving the cone of a chain morphism. Recall that, given a chain morphism $f : (A_*, \partial_A) \to (A'_*, \partial_{A'})$ of degree 0, we define the **cone of $\boldsymbol{f}$** as the chain complex

$$\mathcal{C}(f)_* := A'_{*+1} \oplus A_*,$$

with differential ∂ given in matrix form by

$$\partial := \begin{pmatrix} \partial_{A'} & f \\ 0 & \partial_A \end{pmatrix}.$$

There is a short exact sequence of complexes

$$0 \longrightarrow A'_{*+1} \xrightarrow{i} \mathcal{C}(f)_* \xrightarrow{p} A_* \longrightarrow 0, \tag{5.5}$$

with i, p the obvious inclusion, respectively projection. The main property of the cone construction is that the connecting homomorphism in the homology long exact sequence associated to (5.5) is precisely $f_* : H_*(A) \to H_*(A')$.

Lemma 5.6. *Let*

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r] \arrow[d, "f"] & B_* \arrow[r] \arrow[d, "g"] & C_* \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & C'_* \arrow[r] & 0 \end{tikzcd} \tag{5.6}$$

be a morphism of short exact sequences of complexes. This induces the commutative diagram of homological long exact sequences

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & H_{*+1}(A') \arrow[r] \arrow[d] & H_{*+1}(B') \arrow[r] \arrow[d] & H_{*+1}(C') \arrow[r] \arrow[d] & H_*(A') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{*+1}(\mathcal{C}(f)) \arrow[r] \arrow[d] & H_{*+1}(\mathcal{C}(g)) \arrow[r] \arrow[d] & H_{*+1}(\mathcal{C}(h)) \arrow[r] \arrow[d] & H_*(\mathcal{C}(f)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{*+1}(A) \arrow[r] \arrow[d, "f_*"] & H_{*+1}(B) \arrow[r] \arrow[d, "g_*"] & H_{*+1}(C) \arrow[r] \arrow[d, "h_*"] & H_*(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_*(A') \arrow[r] \arrow[d] & H_*(B') \arrow[r] \arrow[d] & H_*(C') \arrow[r] \arrow[d] & H_{*-1}(A') \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
\tag{5.7}
$$

Proof. Applying the cone construction to each column of (5.6) we obtain the short exact sequence of short exact sequences of complexes

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A'_{*+1} \arrow[r] \arrow[d] & B'_{*+1} \arrow[r] \arrow[d] & C'_{*+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C}(f)_* \arrow[r] \arrow[d] & \mathcal{C}(g)_* \arrow[r] \arrow[d] & \mathcal{C}(h)_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_* \arrow[r] \arrow[d] & B_* \arrow[r] \arrow[d] & C_* \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{5.8}
$$

The lines/columns in (5.7) are obtained as homological long exact sequences associated to the horizontal/vertical short exact sequences in (5.8). Commutativity of the horizontal strips in (5.7) follows from functoriality of the homological long exact sequence with respect to morphisms of short exact sequences of complexes. More precisely, for the first two horizontal strips in (5.7) we use (5.8), and for the third horizontal strip in (5.7) we use (5.6). $\square$

Proof of Theorem 1.2. Given $H \in \mathcal{H}'^{S^1}_{N,\mathrm{reg}}$, $(J,g) \in \mathcal{J}^{S^1}_{N,\mathrm{reg}}(H)$, and a generic collection of perfect Morse functions $f_p : S_p \to \mathbb{R}$, $p \in \mathcal{P}^0(H)$, we denote $C_* := BC^N_*(H, \{f_p\}, J, g)$ (we recall that we work in the trivial free homotopy class). Filtering by the action as in the definition of $SC^{\pm,S^1,N}_*(H,J,g)$ in Section 4.2, we obtain filtered complexes $C^\pm_* := BC^{\pm,N}_*(H,\{f_p\},J,g)$. We denote by $(E^{\pm,N;r}_{*,*}, \bar{d}^r)$ the corresponding spectral sequences, which degenerate at $r=3$ for dimensional reasons.

Since $d^0 = 0$, we have a short exact sequence $0 \to (E^{-;1}_{*,*}, \bar{d}) \to (E^1_{*,*}, \bar{d}) \to (E^{+;1}_{*,*}, \bar{d}) \to 0$ with $\bar{d} = \bar{d}^1 + \bar{d}^2$. This can be rewritten as a morphism of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & (E^{-;1}_{*,0}, \bar{d}^1) \arrow[r] \arrow[d, "\bar{d}^2"] & (E^1_{*,0}, \bar{d}^1) \arrow[r] \arrow[d, "\bar{d}^2"] & (E^{+;1}_{*,0}, \bar{d}^1) \arrow[r] \arrow[d, "\bar{d}^2"] & 0 \\
0 \arrow[r] & (E^{-;1}_{*-2,1}, \bar{d}^1) \arrow[r] & (E^1_{*-2,1}, \bar{d}^1) \arrow[r] & (E^{+;1}_{*-2,1}, \bar{d}^1) \arrow[r] & 0
\end{tikzcd} \tag{5.9}$$

We claim that the commutative diagram (1.7) in the statement is obtained by applying Lemma 5.6 to (5.9). This follows from the following three observations. Firstly, the cone $\mathcal{C}(\bar{d}^2)$ is canonically identified with $(C, \bar{d})$, respectively $(C^\pm, \bar{d})$, so that its homology is $H_*(BC^N_*(H), d)$, respectively $H_*(BC^{\pm,N}_*(H), d)$. Secondly, the homology of $(E^{N;1}_{*,i}, \bar{d}^1)$, $i = 0,1$ is isomorphic to $SH^{S^1,N}_*(H)$, and the homology of $(E^{\pm,N;1}_{*,i}, \bar{d}^1)$, $i=0,1$ is isomorphic to $SH^{\pm,S^1,N}_*(H)$. Thirdly, a straightforward verification shows that, via the above identifications with the cone $\mathcal{C}(\bar{d}^2)$, the Gysin exact sequences obtained from the spectral sequences $E^{N;r}_{*,*}$ and $E^{\pm,N;r}_{*,*}$ coincide with the homological long exact sequences of the corresponding cone constructions.

Passing to the direct limit on $H \in \mathcal{H}'^{S^1}_{N,\mathrm{reg}}$ and $N \to \infty$ we obtain the commutative diagram (1.7). $\square$

Remark 5.7. Denoting the maps in the Gysin exact sequence by

$$\cdots \longrightarrow SH_*(W) \xrightarrow{E} SH^{S^1}_*(W) \xrightarrow{D} SH^{S^1}_{*-2}(W) \xrightarrow{M} SH_{*-1}(W) \longrightarrow \cdots$$

we defined in the Introduction the Batalin-Vilkovisky (BV) operator

$$\Delta := M \circ E : SH_*(W) \to SH_{*+1}(W).$$

The above interpretation of the Gysin exact sequence as the long exact sequence of the cone $C_* := \mathcal{C}(\bar{d}^2)$ allows us to give the following description of

Δ at chain level. We identify $C_* = BC_*^N(H, \{f_p\}, J, g)$ with $SC_{*-1}^{S^1} \oplus SC_*^{S^1} := SC_{*-1}^{S^1,N}(H, J, g) \oplus SC_*^{S^1,N}(H, J, g)$ via

$$m_p \longmapsto (S_p, 0), \qquad M_p \longmapsto (0, S_p).$$

Via this identification, the map Δ is induced by the chain map $\bar{\Delta} : C_* \to C_{*+1}$ given by

$$\bar{\Delta} : SC_{*-1}^{S^1} \oplus SC_*^{S^1} \longrightarrow SC_*^{S^1} \oplus SC_{*+1}^{S^1},$$
$$(S_p, S_q) \longmapsto (S_q, 0).$$

Indeed, the short exact sequence of the cone $\mathcal{C}(\bar{d}^2) = SC_{*-1}^{S^1} \oplus SC_*^{S^1}$ writes

$$0 \to SC_{*-1}^{S^1} \xrightarrow{i} SC_{*-1}^{S^1} \oplus SC_*^{S^1} \xrightarrow{p} SC_*^{S^1} \to 0.$$

The maps i and p are the canonical inclusion and projection. The connecting homomorphism in the homological long exact sequence is the map D, so that the maps i and p induce M and E respectively. Hence the composition $\bar{\Delta} = i \circ p$ induces $\Delta = M \circ E$.

5.2 Filtered continuation maps for parametrized symplectic homology

Let H_s, $s \in \mathbb{R}$ be a smooth increasing homotopy from $H_- \in \mathcal{H}_{N,\text{reg}}^{S^1}$ to $H_+ \in \mathcal{H}_{N,\text{reg}}^{S^1}$, such that $H_s \equiv H_-$ for $s << 0$ and $H_s \equiv H_+$ for $s >> 0$. Let $(J_\pm, g_\pm) \in \mathcal{J}_{N,\text{reg}}^{S^1}(H_\pm)$ and (J_s, g_s), $s \in \mathbb{R}$ a regular smooth homotopy in $\mathcal{J}_N^{S^1}$ from (J_-, g_-) to (J_+, g_+), which is constant near $\pm\infty$. Given $\overline{p} \in \mathcal{P}(H_-)$ and $\underline{p} \in \mathcal{P}(H_+)$, we define the **moduli space of s-dependent S^1-equivariant Floer trajectories** $\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H_s, J_s, g_s)$ to consist of pairs (u, λ) with

$$u : \mathbb{R} \times S^1 \to \widehat{W}, \qquad \lambda : \mathbb{R} \to S^{2N+1}$$

satisfying

$$\partial_s u + J_{s,\lambda(s)}^\theta \partial_\theta u - J_{s,\lambda(s)}^\theta X_{H_{s,\lambda(s)}}^\theta(u) = 0, \tag{5.10}$$

$$\dot{\lambda}(s) - \int_{S^1} \vec{\nabla}_\lambda H_s(\theta, u(s, \theta), \lambda(s)) d\theta = 0, \tag{5.11}$$

and

$$\lim_{s\to-\infty} (u(s, \cdot), \lambda(s)) \in S_{\overline{p}}, \qquad \lim_{s\to+\infty} (u(s, \cdot), \lambda(s)) \in S_{\underline{p}}. \tag{5.12}$$

Due to the s-dependence, the additive group $\mathbb{R}$ does not act on the moduli space $\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H_s, J_s, g_s)$. Recall that, for each $p = (\gamma, \lambda) \in \mathcal{P}(H_\pm)$, we have chosen a cylinder $\sigma_p : [0,1] \times S^1 \to \widehat{W}$ such that $\sigma_p(0, \cdot) = l_{[\gamma]}$ and $\sigma_p(1, \cdot) = \gamma$. We define $\overline{\sigma}_p(s, \theta) := \sigma_p(1-s, \theta)$. We define

$$\mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H_s, J_s, g_s) \subset \mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H_s, J_s, g_s)$$

to consist of trajectories (u, λ) such that $[\sigma_{\overline{p}} \# u \# \overline{\sigma}_{\underline{p}}] = A \in H_2(\widehat{W}; \mathbb{Z})$. It follows from Proposition 4.4 that

$$\dim \mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H_s, J_s, g_s) = -\mu(\overline{p}) + \mu(\underline{p}) + 2\langle c_1(T\widehat{W}), A\rangle + 1. \tag{5.13}$$

For each S^1-orbit of critical points $S_p \subset \mathcal{P}(H_\pm)$ we choose a perfect Morse function $f_p^\pm : S_p \to \mathbb{R}$. We denote by m_p, M_p the minimum, respectively the maximum of $f_p^\pm$. Given $\overline{p} \in \mathcal{P}(H_-)$, $\underline{p} \in \mathcal{P}(H_+)$, $Q_{\overline{p}} \in \mathrm{Crit}(f_{\overline{p}}^-)$, $Q_{\underline{p}} \in \mathrm{Crit}(f_{\underline{p}}^+)$, $A \in H_2(W; \mathbb{Z})$, and $m_\pm \geq 0$, we denote by

$$\mathcal{M}^A_{m_-, m_+}(Q_{\overline{p}}, Q_{\underline{p}}; H_s, \{f_p^\pm\}, J_s, g_s)$$

the union for $p_1^-, \ldots, p_{m_-}^- \in \mathcal{P}(H_-)$, $p_1^+, \ldots, p_{m_+}^+ \in \mathcal{P}(H_+)$, and $A_1^- + \cdots + A_{m_-}^- + A^0 + A_1^+ + \cdots + A_{m_+}^+ = A$ of the fibered products

$$\begin{aligned}
&W^u(Q_{\overline{p}}) \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A_1^-}(S_{\overline{p}}, S_{p_1^-}; H_-, J_-, g_-) \times \mathbb{R}^+) \\
&{}_{\varphi_{f^-_{p_1^-}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \cdots {}_{\varphi_{f^-_{p^-_{m_- -1}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A^-_{m_-}}(S_{p^-_{m_- -1}}, S_{p^-_{m_-}}; H_-, J_-, g_-) \times \mathbb{R}^+) \\
&{}_{\varphi_{f^-_{p^-_{m_-}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A^0}(S_{p^-_{m_-}}, S_{p_1^+}; H_s, J_s, g_s) \times \mathbb{R}^+) \\
&{}_{\varphi_{f^+_{p_1^+}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A_1^+}(S_{p_1^+}, S_{p_2^+}; H_+, J_+, g_+) \times \mathbb{R}^+) \\
&{}_{\varphi_{f^+_{p_2^+}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \cdots {}_{\varphi_{f^+_{p^+_{m_+}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \mathcal{M}^{A^+_{m_+}}(S_{p^+_{m_+}}, S_{\underline{p}})_{\underline{\mathrm{ev}}} \times W^s(Q_{\underline{p}}).
\end{aligned}$$

It follows from [3, Lemma 3.6] that, for a generic choice of the collection of Morse functions $\{f_p^\pm\}$, the previous fibered product is a smooth manifold of dimension

$$\begin{aligned}
&\dim \mathcal{M}^A_{m_-, m_+}(Q_{\overline{p}}, Q_{\underline{p}}; H_s, \{f_p^\pm\}, J_s, g_s) \\
&= -\mu(\overline{p}) + \mathrm{ind}_{f^-_{\overline{p}}}(Q_{\overline{p}}) + \mu(\underline{p}) - \mathrm{ind}_{f^+_{\underline{p}}}(Q_{\underline{p}}) + 2\langle c_1(T\widehat{W}), A\rangle.
\end{aligned}$$

We denote

$$\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H_s, \{f_p^\pm\}, J_s, g_s) := \bigcup_{m_\pm \geq 0} \mathcal{M}^A_{m_-,m_+}(Q_{\overline{p}}, Q_{\underline{p}}; H_s, \{f_p^\pm\}, J_s, g_s).$$

Whenever $\dim \mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H_s, \{f_p^\pm\}, J_s, g_s) = 0$, we can associate a sign $\varepsilon(\mathbf{u})$ to each of its elements via the choice of coherent orientations and the fibered sum rule [3, Section 4.4]. We define the **continuation morphism**

$$\sigma_{H_+,H_-} : BC_*^{a,N}(H_-) \to BC_*^{a,N}(H_+)$$

by

$$\sigma_{H_+,H_-}(Q_{\overline{p}}) := \sum_{\substack{\underline{p}\in\mathcal{P}^a(H_+), Q_{\underline{p}}\in \mathrm{Crit}(f_{\underline{p}}^+) \\ |Q_{\overline{p}}|-|Q_{\underline{p}}e^A|=0}} \ \sum_{\mathbf{u}\in\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H_s, \{f_\gamma^\pm\}, J_s, g_s)} \epsilon(\mathbf{u}) Q_{\underline{p}}\, e^A,$$

for all $\overline{p} \in \mathcal{P}(H_-)$ and $Q_{\overline{p}} \in \mathrm{Crit}(f_{\overline{p}}^-)$. In order to emphasize the homotopy used to define σ_{H_+,H_-}, we shall sometimes write $\sigma_{H_+,H_-}^{H_s}$.

Proof of Lemma 5.5. That the map σ_{H_+,H_-} is a chain morphism satisfying $\sigma_{H_+,H_-} \circ d = d \circ \sigma_{H_+,H_-}$ follows from a straightforward generalization of the Correspondence Theorem 3.7 in [3]. Via the identification of the parametrized Morse-Bott complexes with the Floer complexes of suitable perturbations of the Hamiltonians $H_\pm$, the morphism σ_{H_+,H_-} corresponds to the continuation morphism induced by an increasing homotopy of Hamiltonians.

That σ_{H_+,H_-} preserves the filtration follows from the fact that each of the moduli spaces $\mathcal{M}^{A^0}(S_{p^-_{m_-}}, S_{p_1^+}; H_s, J_s, g_s)$, $\mathcal{M}^{A_j^-}(S_{p^-_{i-1}}, S_{p_i^-}; H_-, J_-, g_-)$, $1 \leq i \leq m_-$ and $\mathcal{M}^{A_i^+}(S_{p_i^+}, S_{p^+_{i+1}}; H_+, J_+, g_+)$, $1 \leq i \leq m_+$ carries a free S^1-action (we denote $p_0^- = \overline{p}$, $p^+_{m_++1} = \underline{p}$). In case they are nonempty, their dimension is therefore at least 1. It then follows from the dimension formulas (5.13) and (4.11) that $|\overline{p}| - |\underline{p}e^A| = -\mu(\overline{p}) + \mu(\underline{p}) + 2\langle c_1(T\widehat{W}), A\rangle \geq 0$. $\square$

For the next statement it is useful to introduce the following algebraic concept. Let (C_*, d_C) and (D_*, d_D) be differential complexes endowed with increasing filtrations $F_\ell C_*$, $F_\ell D_*$, $\ell \in \mathbb{Z}$. A map $K : C_* \to D_*$ is said to be **of order $\boldsymbol{k \geq 0}$** if $K(F_\ell C_*) \subset F_{\ell+k} D_*$ (we allow K to shift the grading). This definition is relevant in the following context. Assume $f, g : C_* \to D_*$ are filtration preserving chain maps such that $f - g = d_D \circ K + K \circ d_C$ for a

chain homotopy $K : C_* \to D_{*+1}$ of order $k \geq 0$. Then the maps f_r, g_r, $r \geq 0$ induced on the associated spectral sequences are homotopic for $r = k$, and coincide for $r > k$ [16, Exercise 3.8, p.87].

Proposition 5.8. *Let $H_- \leq H_+$ be Hamiltonians in $\mathcal{H}^{S^1}_{N,\mathrm{reg}}$ and $H^0_s, H^1_s \in \mathcal{H}^{S^1}_N$, $s \in \mathbb{R}$ be generic smooth increasing homotopies from H_- to H_+, which are constant near $\pm\infty$. Let $(J_\pm, g_\pm) \in \mathcal{J}^{S^1}_{N,\mathrm{reg}}(H_\pm)$ and (J^0_s, g^0_s), (J^1_s, g^1_s) be two generic smooth homotopies in $\mathcal{J}^{S^1}_N$ from (J_-, g_-) to (J_+, g_+), which are constant near $\pm\infty$. A generic homotopy of homotopies $(H^\rho_s, J^\rho_s, g^\rho_s)$, $\rho \in [0,1]$ induces a map $K : BC^{a,N}_*(H_-) \to BC^{a,N}_{*+1}(H_+)$ of order 1 such that*

$$\sigma^{H^1_s}_{H_+,H_-} - \sigma^{H^0_s}_{H_+,H_-} = d \circ K + K \circ d.$$

Proof. Given $\overline{p} \in \mathcal{P}(H_-)$, $\underline{p} \in \mathcal{P}(H_+)$, $Q_{\overline{p}} \in \mathrm{Crit}(f^-_{\overline{p}})$, $Q_{\underline{p}} \in \mathrm{Crit}(f^+_{\underline{p}})$, and $A \in H_2(W;\mathbb{Z})$ such that

$$|Q_{\overline{p}}| - |Q_{\underline{p}} e^A| = 0,$$

we define

$$\mathcal{M}^A := \bigcup_{\rho\in[0,1]} \mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H^\rho_s, \{f^\pm_p\}, J^\rho_s, g^\rho_s).$$

For a generic choice of the triple $(H^\rho_s, J^\rho_s, g^\rho_s)$, $\rho \in [0,1]$, the space $\mathcal{M}^A$ is a smooth 1-dimensional manifold. Its boundary splits as

$$\partial \mathcal{M}^A = \partial^0 \mathcal{M}^A \cup \partial^1 \mathcal{M}^A \cup \partial^{int} \mathcal{M}^A.$$

Here $\partial^i \mathcal{M}^A = \mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H^i_s, \{f^\pm_p\}, J^i_s, g^i_s)$, $i = 0, 1$ and $\partial^{int}\mathcal{M}^A$ corresponds to degeneracies at some point $\rho \in]0,1[$, namely

$$\begin{aligned}
&\partial^{int}\mathcal{M}^A \\
&= \bigcup_{\rho\in]0,1[} \mathcal{M}^B(Q_{\overline{p}}, Q_{p_-}; H_-, \{f^-_p\}, J_-, g_-) \times \mathcal{M}^{A-B}(Q_{p_-}, Q_{\underline{p}}; H^\rho_s, \{f^\pm_p\}, J^\rho_s, g^\rho_s) \\
&\ \cup \bigcup_{\rho\in]0,1[} \mathcal{M}^{A-B}(Q_{\overline{p}}, Q_{p_+}; H^\rho_s, \{f^\pm_p\}, J^\rho_s, g^\rho_s) \times \mathcal{M}^B(Q_{p_+}, Q_{\underline{p}}; H_+, \{f^+_p\}, J_+, g_+).
\end{aligned}$$

Here the union is taken over $B \in H_2(W;\mathbb{Z})$, $p_\pm \in \mathcal{P}(H_\pm)$, $Q_{p_\pm} \in \mathrm{Crit}(f^\pm_{p_\pm})$ such that $|Q_{p_-}| - |Q_{\underline{p}} e^{A-B}| = -1$ and $|Q_{\overline{p}}| - |Q_{p_+} e^{A-B}| = -1$. For a generic choice of the triple $(H^\rho_s, J^\rho_s, g^\rho_s)$, $\rho \in [0,1]$, there are only a finite number of

values of ρ involved in the above union. The elements of $\partial^{int}\mathcal{M}^A$ correspond to the breaking of a gradient trajectory involved in one of the fiber products defining $\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H_s^\rho, \{f_p^\pm\}, J_s^\rho, g_s^\rho)$, as ρ converges to some $\rho_0 \in]0,1[$. There are yet two other types of degeneracy in $\mathcal{M}^A$, which compensate each other: the length of a gradient trajectory in a fibered product as above can shrink to zero, or a Floer trajectory can break at a point $Q \in S_p \setminus \mathrm{Crit}(f_p^\pm)$, for some $p \in \mathcal{P}(H_\pm)$.

We define $K : BC_*^{a,N}(H_-) \to BC_{*+1}^{a,N}(H_+)$ by

$$K(Q_{\overline{p}}) = \sum_{\rho\in]0,1[} \; \sum_{|Q_{\overline{p}}|-|Q_{\underline{p}}e^A|=-1} \; \sum_{\mathbf{u}\in\mathcal{M}^A(Q_{\overline{p}},Q_{\underline{p}};H_s^\rho,\{f_p^\pm\},J_s^\rho,g_s^\rho)} \varepsilon(\mathbf{u})Q_{\underline{p}}e^A.$$

The above description of $\partial\mathcal{M}^A$ shows that we have indeed $\sigma_{H_+,H_-}^{H_s^1} - \sigma_{H_+,H_-}^{H_s^0} = d\circ K + K \circ d$. That the chain homotopy K is of order 1 means that it satisfies $K(F_\ell B_*^{a,N}(H_-)) \subset F_{\ell+1}B_{*+1}^{a,N}(H_+)$. This follows from the fact that each family of moduli spaces $\bigcup_{\rho\in]0,1[}\mathcal{M}^{A^0}(Q_{p_-}, Q_{p_+}; H_s^\rho, \{f_p^\pm\}, J_s^\rho, g_s^\rho)$ carries a free S^1-action. In case it is nonempty, its dimension must therefore be at least 1. On the other hand, it follows from (5.13) that this dimension is equal to $|p_-| - |p_+e^{A^0}| + 2$, so that $|p_-| - |p_+e^{A^0}| \geq -1$. A similar argument shows that, for the moduli spaces $\mathcal{M}^{A^\pm}(Q_{p_0^\pm}, Q_{p_1^\pm}; H_\pm, \{f_p^\pm\}, J_\pm, g_\pm)$ appearing in the definition of K, we must have $|p_0^\pm| - |p_1^\pm e^{A^\pm}| \geq 0$. Thus, for the fibered products appearing in the definition of K we have $|\overline{p}| - |\underline{p}e^A| \geq -1$. $\square$

Proposition 5.9. *Let $H_0 \leq H_1 \leq H_2$ be three Hamiltonians in $\mathcal{H}_{N,\mathrm{reg}}^{S^1}$, and let $H_s^{01}, H_s^{12} \in \mathcal{H}_N^{S^1}$, $s \in \mathbb{R}$ be two generic smooth increasing homotopies from H_0 to H_1, respectively from H_1 to H_2, which are constant near $\pm\infty$. Let $(J_i, g_i) \in \mathcal{J}_{N,\mathrm{reg}}^{S^1}(H_i)$, $i = 0, 1, 2$ and (J_s^{01}, g_s^{01}), (J_s^{12}, g_s^{12}) be two generic smooth homotopies in $\mathcal{J}_N^{S^1}$ from (J_0, g_0) to (J_1, g_1), respectively from (J_1, g_1) to (J_2, g_2), which are constant near $\pm\infty$. For $R > 0$ sufficiently large we denote*

$$H_s^{02,R} := \begin{cases} H_{s+R}^{01}, & s \leq 0, \\ H_{s-R}^{12}, & s \geq 0. \end{cases}$$

We define the homotopies $J_s^{02,R}, g_s^{02,R}$ in a similar way. There exists a map $K : BC_*^{a,N}(H_0) \to BC_{*+1}^{a,N}(H_2)$ of order 1 such that

$$\sigma_{H_2,H_1}^{H_s^{12}} \circ \sigma_{H_1,H_0}^{H_s^{01}} - \sigma_{H_2,H_0}^{H_s^{02,R}} = d \circ K + K \circ d.$$

Proof. Let $\{f^i_p\}$, $i = 0, 1, 2$ be three generic collections of perfect Morse functions on S_p, for $p \in \mathcal{P}(H_i)$ respectively. Given $\overline{p} \in \mathcal{P}(H_0)$, $\underline{p} \in \mathcal{P}(H_2)$, $Q_{\overline{p}} \in \mathrm{Crit}(f^0_{\overline{p}})$, $Q_{\underline{p}} \in \mathrm{Crit}(f^2_{\underline{p}})$, $A \in H_2(W; \mathbb{Z})$ such that $|Q_{\overline{p}}| - |Q_{\underline{p}} e^A| = 0$, we define for $R_0 > 0$ sufficiently large the family of moduli spaces

$$\mathcal{M}^A_1 := \bigcup_{R \geq R_0} \mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H^{02,R}_s, \{f^0_p, f^2_p\}, J^{02,R}_s, g^{02,R}_s).$$

For a generic choice of the homotopies, this is a smooth 1-dimensional manifold. Its boundary splits as

$$\partial \mathcal{M}^A_1 = \partial^{R_0} \mathcal{M}^A_1 \cup \partial^\infty \mathcal{M}^A_1 \cup \partial^{int} \mathcal{M}^A_1.$$

Here $\partial^{R_0} \mathcal{M}^A_1 = \mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H^{02,R_0}_s, \{f^0_p, f^2_p\}, J^{02,R_0}_s, g^{02,R_0}_s)$. We now describe $\partial^\infty \mathcal{M}^A_1$, which corresponds to degenerations as $R \to \infty$. Let $p \in \mathcal{P}(H_1)$, $m_0 \geq 0$, $B \in H_2(W; \mathbb{Z})$, and define $\mathcal{M}^B_{m_0}(Q_{\overline{p}}, S_p; H^{01}_s, \{f^0_p\}, J^{01}_s, g^{01}_s)$ as the union for $p^0_1, \ldots, p^0_{m_0} \in \mathcal{P}(H_0)$ and $A^0_1 + \cdots + A^0_{m_0} + A^{01} = B$ of the fibered products

$$\begin{aligned}
& W^u(Q_{\overline{p}}) \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A^0_1}(S_{\overline{p}}, S_{p^0_1}; H_0, J_0, g_0) \times \mathbb{R}^+) \\
& {}_{\varphi_{f^0_{p^0_1}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \cdots {}_{\varphi_{f^0_{p^0_{m_0-1}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A^0_{m_0}}(S_{p^0_{m_0-1}}, S_{p^0_{m_0}}; H_0, J_0, g_0) \times \mathbb{R}^+) \\
& {}_{\varphi_{f^0_{p^0_{m_0}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \mathcal{M}^{A^{01}}(S_{p^0_{m_0}}, S_p; H^{01}_s, J^{01}_s, g^{01}_s).
\end{aligned}$$

We define $\mathcal{M}^B(Q_{\overline{p}}, S_p; H^{01}_s, \{f^0_p\}, J^{01}_s, g^{01}_s)$ as the union over $m_0 \geq 0$ of the moduli spaces $\mathcal{M}^B_{m_0}(Q_{\overline{p}}, S_p; H^{01}_s, \{f^0_p\}, J^{01}_s, g^{01}_s)$. This is a smooth manifold of dimension

$$\dim \mathcal{M}^B(Q_{\overline{p}}, S_p; H^{01}_s, \{f^0_p\}, J^{01}_s, g^{01}_s) = |Q_{\overline{p}}| - |p e^B|.$$

Given $p \in \mathcal{P}(H_1)$, $m_2 \geq 0$, $B \in H_2(W; \mathbb{Z})$, we define the moduli space $\mathcal{M}^B_{m_2}(S_p, Q_{\underline{p}}; H^{12}_s, \{f^2_p\}, J^{12}_s, g^{12}_s)$ as the union for $p^2_1, \ldots, p^2_{m_2} \in \mathcal{P}(H_2)$ and $A^{12} + A^2_1 + \cdots + A^2_{m_2} = B$ of the fibered products

$$\begin{aligned}
& (\mathcal{M}^{A^{12}}(S_p, S_{p^2_1}; H^{12}_s, J^{12}_s, g^{12}_s) \times \mathbb{R}^+) \\
& {}_{\varphi_{f^2_{p^2_1}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A^2_1}(S_{p^2_1}, S_{p^2_2}; H_2, J_2, g_2) \times \mathbb{R}^+) \\
& {}_{\varphi_{f^2_{p^2_2}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \cdots {}_{\varphi_{f^2_{p^2_{m_2}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \mathcal{M}^{A^2_{m_2}}(S_{p^2_{m_2}}, S_{\underline{p}})_{\underline{\mathrm{ev}}} \times W^s(Q_{\underline{p}}).
\end{aligned}$$

We define $\mathcal{M}^B(S_p, Q_{\underline{p}}; H_s^{12}, \{f_p^2\}, J_s^{12}, g_s^{12})$ as the union over $m_2 \geq 0$ of the moduli spaces $\mathcal{M}^B_{m_2}(S_p, Q_{\underline{p}}; H_s^{12}, \{f_p^2\}, J_s^{12}, g_s^{12})$. This is a smooth manifold of dimension

$$\dim \mathcal{M}^B(S_p, Q_{\underline{p}}; H_s^{12}, \{f_p^2\}, J_s^{12}, g_s^{12}) = |p| - |Q_{\underline{p}} e^B| + 1.$$

The boundary $\partial^\infty \mathcal{M}_1^A$ is then equal to

$$\bigcup_{\substack{p\in\mathcal{P}(H_1)\\ B_0+B_2=A}} \mathcal{M}^{B_0}(Q_{\overline{p}}, S_p; H_s^{01}, \{f_p^0\}, J_s^{01}, g_s^{01})_{\underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \mathcal{M}^{B_2}(S_p, Q_{\underline{p}}; H_s^{12}, \{f_p^2\}, J_s^{12}, g_s^{12}).$$

The boundary $\partial^{int}\mathcal{M}_1^A$ corresponds to degeneracies at a point $R \in]R_0, \infty[$, namely

$$\begin{aligned}
&\partial^{int}\mathcal{M}_1^A \\
&= \bigcup_{R>R_0} \mathcal{M}^{B_0}(Q_{\overline{p}}, Q_{p_0}; H_0, \{f_p^0\}, J_0, g_0) \times \mathcal{M}^{B_2}(Q_{p_0}, Q_{\underline{p}}; H_s^{02,R}, \{f_p^i\}, J_s^{02,R}, g_s^{02,R}) \\
&\cup \bigcup_{R>R_0} \mathcal{M}^{B_0}(Q_{\overline{p}}, Q_{p_2}; H_s^{02,R}, \{f_p^i\}, J_s^{02,R}, g_s^{02,R}) \times \mathcal{M}^{B_2}(Q_{p_2}, Q_{\underline{p}}; H_2, \{f_p^2\}, J_2, g_2).
\end{aligned}$$

Here we used the shortcut notation $\{f_p^i\} = \{f_p^0, f_p^2\}$, and the union is taken over $B_0 + B_2 = A$, $p_i \in \mathcal{P}(H_i)$, $Q_{p_i} \in \mathrm{Crit}(f^i_{p_i})$, $i = 0, 2$, such that $|Q_{p_0}| - |Q_{\underline{p}} e^{A-B}| = -1$ and $|Q_{\overline{p}}| - |Q_{p_2} e^{A-B}| = -1$. For a generic choice of the triple $(H_s^{02,R}, J_s^{02,R}, g_s^{02,R})$, $R \geq R_0$, there are only a finite number of values of R involved in the above union. The elements of $\partial^{int}\mathcal{M}_1^A$ correspond to the breaking of a gradient trajectory involved in one of the fiber products defining $\mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H_s^{02,R}, \{f_p^0, f_p^2\}, J_s^{02,R}, g_s^{02,R})$, as R converges to some $R_{int} \in]R_0, \infty[$. There are yet two other types of degeneracy in $\mathcal{M}_1^A$, which compensate each other: the length of a gradient trajectory in a fibered product as above can shrink to zero, or a Floer trajectory can break at a point $Q \in S_p \setminus \mathrm{Crit}(f_p^i)$, for some $p \in \mathcal{P}(H_i)$, $i = 0, 2$. We define a map $K_1 : BC_*^{a,N}(H_0) \to BC_{*+1}^{a,N}(H_2)$ by

$$K_1(Q_{\overline{p}}) = \sum_{R>R_0} \sum_{|Q_{\overline{p}}| - |Q_{\underline{p}} e^A| = -1} \sum_{\mathbf{u} \in \mathcal{M}^A(Q_{\overline{p}}, Q_{\underline{p}}; H_s^{02,R}, \{f_p^0, f_p^2\}, J_s^{02,R}, g_s^{02,R})} \varepsilon(\mathbf{u}) Q_{\underline{p}} e^A.$$

The same argument as in the proof of Proposition 5.8 shows that K_1 is of order 1. The previous description of $\partial \mathcal{M}_1^A$ can be summarized by saying that

$d\circ K_1 + K_1\circ d + \sigma_{H_2,H_0}^{H_s^{02,R_0}}$ is equal to the chain map obtained by the count of elements in $\partial^\infty\mathcal{M}_1^A$.

We now exhibit another 1-dimensional moduli space whose boundary contains $\partial^\infty\mathcal{M}_1^A$. Given $\overline{p}\in\mathcal{P}(H_0)$, $\underline{p}\in\mathcal{P}(H_2)$, $Q_{\overline{p}}\in \mathrm{Crit}(f^0_{\overline{p}})$, $Q_{\underline{p}}\in\mathrm{Crit}(f^2_{\underline{p}})$, $A\in H_2(W;\mathbb{Z})$, and $m_0,m_1,m_2\geq 0$, we denote by

$$\mathcal{M}^A_{m_0,m_1,m_2}(Q_{\overline{p}},Q_{\underline{p}};H_s^{ij},\{f_p^k\},J_s^{ij},g_s^{ij})$$

the union for $p^0_1,\dots,p^0_{m_0}\in\mathcal{P}(H_0)$, $p^1_1,\dots,p^1_{m_1+1}\in\mathcal{P}(H_1)$, $p^2_1,\dots,p^2_{m_2}\in\mathcal{P}(H_2)$, and $A^0_1+\cdots+A^0_{m_0}+A^{01}+A^1_1+\cdots+A^1_{m_1}+A^{12}+A^2_1+\cdots+A^2_{m_2}=A$ of the fibered products

$$\begin{aligned}
&W^u(Q_{\overline{p}})\times_{\overline{\mathrm{ev}}}(\mathcal{M}^{A^0_1}(S_{\overline{p}},S_{p^0_1};H_0,J_0,g_0)\times\mathbb{R}^+)\\
&{}_{\varphi_{f^0_{p^0_1}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}\cdots{}_{\varphi_{f^0_{p^0_{m_0-1}}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}(\mathcal{M}^{A^0_{m_0}}(S_{p^0_{m_0-1}},S_{p^0_{m_0}};H_0,J_0,g_0)\times\mathbb{R}^+)\\
&{}_{\varphi_{f^0_{p^0_{m_0}}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}(\mathcal{M}^{A^{01}}(S_{p^0_{m_0}},S_{p^1_1};H_s^{01},J_s^{01},g_s^{01})\times\mathbb{R}^+)\\
&{}_{\varphi_{f^1_{p^1_1}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}(\mathcal{M}^{A^1_1}(S_{p^1_1},S_{p^1_1};H_1,J_1,g_1)\times\mathbb{R}^+)\\
&{}_{\varphi_{f^1_{p^1_1}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}\cdots{}_{\varphi_{f^1_{p^1_{m_1}}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}(\mathcal{M}^{A^1_{m_1}}(S_{p^1_{m_1}},S_{p^1_{m_1+1}};H_1,J_1,g_1)\times\mathbb{R}^+)\\
&{}_{\varphi_{f^1_{p^1_{m_1+1}}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}(\mathcal{M}^{A^{12}}(S_{p^1_{m_1+1}},S_{p^2_1};H_s^{12},J_s^{12},g_s^{12})\times\mathbb{R}^+)\\
&{}_{\varphi_{f^2_{p^2_1}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}(\mathcal{M}^{A^2_1}(S_{p^2_1},S_{p^2_2};H_2,J_2,g_2)\times\mathbb{R}^+)\\
&{}_{\varphi_{f^2_{p^2_2}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}\cdots{}_{\varphi_{f^2_{p^2_{m_2}}}\circ\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}}\mathcal{M}^{A^2_{m_2}}(S_{p^2_{m_2}},S_{\underline{p}})_{\underline{\mathrm{ev}}}\times W^s(Q_{\underline{p}}).
\end{aligned}$$

In the above notation we abridged $H_s^{ij}=\{H_s^{01},H_s^{12}\}$ (similarly for J_s^{ij},g_s^{ij}) and $\{f_p^k\}=\{f_p^0,f_p^1,f_p^2\}$. We denote $\mathcal{M}_2^A:=\mathcal{M}^A(Q_{\overline{p}},Q_{\underline{p}};H_s^{ij},\{f_p^k\},J_s^{ij},g_s^{ij})$ the union over $m_k\geq 0$, $k=0,1,2$ of the previously defined moduli spaces $\mathcal{M}^A_{m_0,m_1,m_2}(Q_{\overline{p}},Q_{\underline{p}};H_s^{ij},\{f_p^k\},J_s^{ij},g_s^{ij})$. This is a smooth manifold of dimension $|Q_{\overline{p}}|-|Q_{\underline{p}}e^A|+1$. In the case $|Q_{\overline{p}}|-|Q_{\underline{p}}e^A|=0$ that we are considering, $\mathcal{M}_2^A$ is a smooth 1-dimensional manifold whose boundary splits as

$$\partial\mathcal{M}_2^A=\partial^0\mathcal{M}_2^A\cup\partial^{\infty,0}\mathcal{M}_2^A\cup\partial^{\infty,1}\mathcal{M}_2^A\cup\partial^{\infty,2}\mathcal{M}_2^A.$$

Here $\partial^0\mathcal{M}_2^A$ corresponds to $m_0=0$ and the length of the gradient trajectory running between the endpoints of the two s-dependent Floer trajectories

being equal to 0. Thus $\partial^0\mathcal{M}_2^A = \partial^\infty\mathcal{M}_1^A$. The elements of $\partial^{\infty,k}\mathcal{M}_2^A$, $k = 0,1,2$ correspond to the breaking of a gradient trajectory of f_p^k appearing in the fibered product which defines $\mathcal{M}_2^A$, for some $p\in\mathcal{P}(H_k)$. Thus we have

$$\partial^{\infty,1}\mathcal{M}_2^A = \bigcup_{\substack{p\in\mathcal{P}(H_1)\\ Q_p\in\mathrm{Crit}(f_p^1)\\ B^{01}+B^{12}=A}} \mathcal{M}^{B^{01}}(Q_{\overline{p}}, Q_p; H_s^{01}, \{f_p^i\}, J_s^{01}, g_s^{01}) \times \mathcal{M}^{B^{12}}(Q_p, Q_{\underline{p}}; H_s^{12}, \{f_p^j\}, J_s^{12}, g_s^{12}).$$

Here we abridged $\{f_p^i\} = \{f_p^0, f_p^1\}$ and $\{f_p^j\} = \{f_p^1, f_p^2\}$. Similarly, we have

$$\partial^{\infty,0}\mathcal{M}_2^A = \bigcup_{\substack{p\in\mathcal{P}(H_0)\\ Q_p\in\mathrm{Crit}(f_p^0)\\ B^0+B^{02}=A}} \mathcal{M}^{B_0}(Q_{\overline{p}}, Q_p; H_0, \{f_p^0\}, J_0, g_0) \times \mathcal{M}^{B^{02}}(Q_p, Q_{\underline{p}}; H_s^{ij}, \{f_p^k\}, J_s^{ij}, g_s^{ij})$$

with $|Q_p| - |Q_{\underline{p}}e^{B^{02}}| = -1$, and

$$\partial^{\infty,1}\mathcal{M}_2^A = \bigcup_{\substack{p\in\mathcal{P}(H_2)\\ Q_p\in\mathrm{Crit}(f_p^2)\\ B^{02}+B^2=A}} \mathcal{M}^{B^{02}}(Q_{\overline{p}}, Q_p; H_s^{ij}, \{f_p^k\}, J_s^{ij}, g_s^{ij}) \times \mathcal{M}^{B^2}(Q_p, Q_{\underline{p}}; H_2, \{f_p^2\}, J_2, g_2)$$

with $|Q_{\overline{p}}| - |Q_p e^{B^{02}}| = -1$. We define $K_2 : BC_*^{a,N}(H_0) \to BC_{*+1}^{a,N}(H_2)$ by

$$K_2(Q_{\overline{p}}) = \sum_{\substack{\underline{p}\in\mathcal{P}(H_2)\\ B^{02}\in H_2(W;\mathbb{Z})\\ |Q_{\overline{p}}|-|Q_{\underline{p}}e^{B^{02}}|=-1}} \quad \sum_{\mathbf{u}\in\mathcal{M}^{B^{02}}(Q_{\overline{p}},Q_{\underline{p}};H_s^{ij},\{f_p^k\},J_s^{ij},g_s^{ij})} \varepsilon(\mathbf{u}) Q_{\underline{p}} e^{B^{02}}.$$

The same argument as in the proof of Proposition 5.8 shows that K_2 is of order 1. The previous description of $\partial\mathcal{M}_2^A$ shows that the chain map determined by the count of elements in $\partial^0\mathcal{M}_2^A$ is equal to $\sigma_{H_2,H_1}^{H_s^{12}} \circ \sigma_{H_1,H_0}^{H_s^{01}} - d\circ K_2 - K_2\circ d$. Since $\partial^0\mathcal{M}_2^A = \partial^\infty\mathcal{M}_1^A$, we obtain the conclusion of the Proposition by setting $K := K_1 + K_2$. □

Proof of Proposition 4.6. We consider a generic homotopy (J_s^{12}, g_s^{12}), $s\in\mathbb{R}$ inside $\mathcal{J}_N^{S^1}$ from (J_1, g_1) to (J_2, g_2), which is constant near $\pm\infty$. Then

$(J^{21}_s, g^{21}_s) := (J^{12}_{-s}, g^{12}_{-s})$ is a homotopy from (J_2, g_2) to (J_1, g_1). These determine filtered chain maps $\sigma_{21} : BC^{a,N}_*(H, J_1, g_1) \to BC^{a,N}_*(H, J_2, g_2)$ and $\sigma_{12} : BC^{a,N}_*(H, J_2, g_2) \to BC^{a,N}_*(H, J_1, g_1)$. By Proposition 5.9, the composition $\sigma_{21} \circ \sigma_{12}$ is homotopic to the filtered chain map $\sigma_{22} : BC^{a,N}_*(H, J_2, g_2) \to BC^{a,N}_*(H, J_2, g_2)$ determined by the concatenation $(J^{21}_s \#_R J^{12}_s, g^{21}_s \#_R g^{12}_s)$ for $R > 0$ large enough. The latter is homotopic to the identity by Proposition 5.8.

Since all the homotopies involved are of order 1, we obtain that $\sigma_{21} \circ \sigma_{12}$ induces on the first page $E^{a,N;1}_{*,*}(H, J_2, g_2)$ of the corresponding spectral sequence a chain morphism which is homotopic to the identity. The induced morphism on the second page $E^{a,N;2}_{*,*}(H, J_2, g_2)$ is therefore the identity. Similarly, $\sigma_{12} \circ \sigma_{21}$ induces the identity on the second page $E^{a,N;2}_{*,*}(H, J_1, g_1)$.

Thus the induced morphism $\sigma_{21} : E^{a,N;2}_{*,*}(H, J_1, g_1) \to E^{a,N;1}_{*,*}(H, J_2, g_2)$ is an isomorphism. Since σ_{21} preserves both the degree and the filtration, it follows that $\sigma_{21}(E^{a,N;2}_{*,1}(H, J_1, g_1)) = E^{a,N;1}_{*,*}(H, J_2, g_2)$. Since $E^{a,N;2}_{*,1}(H, J_i, g_i) \simeq SH^{a,S^1,N}_{*+1}(H, J_i, g_i)$, $i = 1, 2$, we obtain the desired isomorphism. The fact that it does not depend on the choice of homotopy (J^{12}_s, g^{12}_s) is a consequence of Proposition 5.8. □

Remark 5.10. The isomorphism $SH^{a,S^1,N}_*(H, J_1, g_1) \xrightarrow{\simeq} SH^{a,S^1,N}_*(H, J_2, g_2)$ constructed in the proof of Proposition 4.6 is induced by the chain map $SC^{a,S^1,N}_*(H, J_1, g_1) \to SC^{a,S^1,N}_*(H, J_2, g_2)$ given by the count of the elements of the 0-dimensional moduli spaces

$$\mathcal{M}^A_{S^1}(S_{\overline{p}}, S_{\underline{p}}; H, J^{12}_s, g^{12}_s) := \mathcal{M}^A(S_{\overline{p}}, S_{\underline{p}}; H, J^{12}_s, g^{12}_s)/S^1.$$

A The parametrized Robbin-Salamon index

We summarize in this section some important properties of the parametrized Robbin-Salamon index. We recall from §3.2 the definition of the subgroup $\mathcal{S}_{n,m} \subset \mathrm{Sp}(2n + 2m)$, consisting of matrices of the form

$$M = M(\Psi, X, E) = \begin{pmatrix} \Psi & \Psi X & 0 \\ 0 & \mathbb{1} & 0 \\ X^T J_0 & E + \frac{1}{2} X^T J_0 X & \mathbb{1} \end{pmatrix},$$

with $\Psi \in \mathrm{Sp}(2n)$, $X \in \mathrm{Mat}_{2n,m}(\mathbb{R})$, and $E \in \mathrm{Mat}_m(\mathbb{R})$ symmetric. We have denoted by $J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ the standard complex structure on $\mathbb{R}^{2n}$, so that

$\Psi \in \mathrm{Sp}(2n)$ if and only if $\Psi^T J_0 \Psi = J_0$. The standard complex structure on $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$ is

$$\widetilde{J}_0 := \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & -\mathbb{1} \\ 0 & \mathbb{1} & 0 \end{pmatrix},$$

and we have $M^T \widetilde{J}_0 M = \widetilde{J}_0$. Note that we have natural embeddings (which respect the group structure)

$$\mathcal{S}_{n,m} \times \mathcal{S}_{n',m'} \hookrightarrow \mathcal{S}_{n+n',m+m'}$$

which associate to $M = M(\Psi, X, E) \in \mathcal{S}_{n,m}$ and $M' = M(\Psi', X', E') \in \mathcal{S}_{n',m'}$ the matrix

$$M \oplus M' := M(\Psi \oplus \Psi', X \oplus X', E \oplus E') \in \mathcal{S}_{n+n',m+m'}.$$

The space $\mathcal{S}_{n,m}$ is stratified as $\coprod_{k=0}^{2n+m} \mathcal{S}_{n,m}^k$, with

$$\mathcal{S}_{n,m}^k := \big\{ M \in \mathcal{S}_{n,m} \ : \ \dim \ker (M - \mathbb{1}) = m + k. \big\}.$$

Proposition A.1. *The Robbin-Salamon index $\mu = \mu_{RS}$ defined on paths $M : [a,b] \to \mathcal{S}_{n,m}$, $M(\theta) = M(\Psi(\theta), X(\theta), E(\theta))$ has the following properties.*

(Homotopy) *If $M, M' : [a,b] \to \mathcal{S}_{n,m}$ are homotopic with fixed endpoints then*

$$\mu(M) = \mu(M');$$

(Catenation) *For any $c \in [a,b]$ we have*

$$\mu(M) = \mu(M|_{[a,c]}) + \mu(M|_{[c,b]});$$

(Naturality) *For any path $P : [a,b] \to \mathrm{Sp}(2n) \times \mathrm{Sp}(2m)$ of the form*

$$P(\theta) = \begin{pmatrix} \Phi(\theta) & 0 & 0 \\ 0 & A(\theta) & 0 \\ 0 & 0 & A(\theta) \end{pmatrix} \tag{A.1}$$

(with $\Phi(\theta) \in \mathrm{Sp}(2n)$ and $A(\theta) \in \mathrm{O}(m)$), we have

$$\mu(PMP^{-1}) = \mu(M);$$

(Loop) *For any loop* $P : [a,b] \to \mathrm{Sp}(2n) \times \mathrm{Sp}(2m)$ *of the form* (A.1), *we have*

$$\mu(PM) = \mu(M) + 2\mu(\Phi);$$

(Product) *For any* $M \in \mathcal{S}_{n,m}$ *and* $M' \in \mathcal{S}_{n',m'}$ *we have*

$$\mu(M \oplus M') = \mu(M) + \mu(M');$$

(Splitting) *Given* $M = M(\Psi, 0, E) : [a,b] \to \mathcal{S}_{n,m}$, *we have*

$$\mu(M) = \mu(\Psi) + \frac{1}{2}\mathrm{sign}\, E(b) - \frac{1}{2}\mathrm{sign}\, E(a);$$

(Signature) *Given symmetric matrices* $E \in \mathbb{R}^{m\times m}$ *and* $S \in \mathbb{R}^{2n\times 2n}$ *with* $\|S\| < 2\pi$, *we have*

$$\mu\left\{ \begin{pmatrix} \exp(J_0 St) & 0 \\ 0 & tE \end{pmatrix},\ t \in [0,1] \right\} = \frac{1}{2}\mathrm{sign}(S) + \frac{1}{2}\mathrm{sign}(E);$$

(Zero) *For any path* $M : [a,b] \to \mathcal{S}^k_{n,m}$ *we have*

$$\mu(M) = 0;$$

(Integrality) *Given a path* $M : [a,b] \to \mathcal{S}_{n,m}$ *with* $M(a) \in \mathcal{S}^{k_a}_{n,m}$, $M(b) \in \mathcal{S}^{k_b}_{n,m}$, *we have*

$$\mu(M) + \frac{k_a - k_b}{2} \in \mathbb{Z};$$

(Determinant) *Given a path* $M = M(\Psi, X, E) : [a,b] \to \mathcal{S}_{n,m}$ *with* $M(a) = \mathbb{1}$ *and* $M(b) \in \mathcal{S}^0_{n,m}$, *we have*

$$(-1)^{n+\frac{m}{2}-\mu(M)} = \mathrm{sign}\ \det \begin{pmatrix} \Psi - \mathbb{1} & \Psi X \\ X^T J_0 & E + \frac{1}{2}X^T J_0 X \end{pmatrix}.$$

We have denoted for simplicity $\Psi = \Psi(b)$, $X = X(b)$, $E = E(b)$.

(Involution) *For any* $M = M(\Psi, X, E) : [a,b] \to \mathcal{S}_{n,m}$ *we have*

$$\mu(M(\Psi, X, E)) = \mu(M(\Psi, -X, E))$$

and

$$\mu(M(\Psi^{-1}, X, -E)) = \mu(M(\Psi^T, J_0 X, -E)) = -\mu(M(\Psi, X, E)).$$

Proof. The *(Homotopy)*, *(Catenation)*, *(Naturality)*, *(Product)*, and *(Zero)* properties are exactly the corresponding properties of the Robbin-Salamon index [20, Theorem 4.1]. For the *(Naturality)* property, a straightforward verification shows that $PMP^{-1} \in \mathcal{S}_{n,m}$.

To prove the *(Loop)* property we use the equality

$$\mu(PM) = \mu(M) + 2\mu(P) = \mu(M) + 2\mu(\Phi) + 2\mu \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Since $\pi_1(\mathrm{O}(m)) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(\mathrm{Sp}(2m)) = \mathbb{Z}$, the last term vanishes.

The *(Splitting)* property follows from the *(Product)* property and the normalization axiom for the Robbin-Salamon index of a symplectic shear.

The *(Signature)* property follows from the *(Splitting)* property and from the identity $\mu_{RS}(\exp(J_0St)) = \frac{1}{2}\mathrm{sign}(S)$ [24, Theorem 3.3.(iv)].

The *(Integrality)* property follows directly from the analogous property for the Robbin-Salamon index [20, Theorem 4.7].

We prove the *(Involution)* property. The identity $\mu(M(\Psi^{-1}, X, -E)) = \mu(M(\Psi^T, J_0X, -E))$ follows from the *(Naturality)* axiom by conjugating with the constant path $J_0 \oplus \mathbb{1}_{2m}$. The identity $\widetilde{\mu}(M(\Psi, X, E)) = \widetilde{\mu}(M(\Psi, -X, E))$ follows by conjugating twice with $J_0 \oplus \mathbb{1}_{2m}$.

It remains to prove the *(Determinant)* property. Given a path $N : [0, 1] \to \mathrm{Sp}(2n + 2m)$ satisfying $N(0) = \mathbb{1}$ and $\det(N(1) - \mathbb{1}) \neq 0$, we have [24, Theorem 3.3.(iii)]

$$(-1)^{n+m-\mu_{RS}(N)} = \mathrm{sign}\ \det(N(1) - \mathbb{1}).$$

We construct such a path $N : [a, b + \varepsilon] \to \mathrm{Sp}(2n + 2m)$ by catenating $M = M(\Psi(\theta), X(\theta), E(\theta))$ with the path $M' : [b, b + \varepsilon] \to \mathrm{Sp}(2n + 2m)$ given by

$$M'(b + \theta) := \begin{pmatrix} \Psi & \Psi X & \theta\Psi X \\ 0 & \mathbb{1} & \theta\mathbb{1} \\ X^T J_0 & E + \frac{1}{2}X^T J_0 X & \mathbb{1} + \theta(E + \frac{1}{2}X^T J_0 X) \end{pmatrix}.$$

We have denoted for simplicity $\Psi := \Psi(b)$, $X := X(b)$, and $E := E(b)$. Since $M(b) \in \mathcal{S}^0_{n,m}$, the path M' has a single crossing at b and the kernel of $M'(b) - \mathbb{1} = M(b) - \mathbb{1}$ is $\{0\} \oplus \{0\} \oplus \mathbb{R}^m$. The crossing form at b is $-\mathbb{1}_m$, so that $\mu_{RS}(M') = -\frac{m}{2}$. Thus $\mu_{RS}(N) = \mu(M) - \frac{m}{2}$. On the other hand

$$\det(N(b + \varepsilon) - \mathbb{1}) = \varepsilon^m(-1)^m \det \begin{pmatrix} \Psi - \mathbb{1} & \Psi X \\ X^T J_0 & E + \frac{1}{2}X^T J_0 X \end{pmatrix}.$$

This implies the desired statement. □

Example A.2. The index $\mu(M(\Psi, X, E))$ depends in an essential way on X, as the following example shows. Given $a, b \in \mathbb{R}$, let

$$\Psi := \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad X_{a,b} := \begin{pmatrix} a \\ b \end{pmatrix}, \qquad E := 1.$$

We denote $M_{a,b} := M(\Psi, X_{a,b}, E) \in \mathcal{S}_{1,1}$. It follows from the *(Determinant)* property that a path in $\mathcal{S}_{1,1}$ starting at $\mathbb{1}$ and ending at $M_{0,0}$ has an index in $\frac{1}{2} + 2\mathbb{Z}$, whereas a path in $\mathcal{S}_{1,1}$ starting at $\mathbb{1}$ and ending at $M_{1,1}$ has an index in $\frac{1}{2} + 2\mathbb{Z} + 1$ (the value of the relevant determinant is $-\frac{1}{2} + \frac{3}{2}ab$).

For the rest of this Appendix we place ourselves in $\mathbb{R}^{2N}$ equipped with the standard symplectic form ω_0 and the standard complex structure J_0. The next Proposition is relevant for the parametrized Robbin-Salamon index when applied with $N = n + m$ and $E(t) \equiv \{0\} \oplus \{0\} \oplus \mathbb{R}^m$. We recall that, given a path of symplectic matrices $M : [0, 1] \to \mathrm{Sp}(2N)$, the crossing form at a point $t \in [0, 1]$ is the quadratic form $\Gamma(M, t)$ on $\ker(M(t) - \mathbb{1})$ given by $\Gamma(M, t)(v) = \langle v, -J_0\dot{M}(t)M(t)^{-1}v\rangle$.

Proposition A.3. *Let $M : [0, 1] \to \mathrm{Sp}(2N)$ be a C^1-path of symplectic matrices with the following property: there exists a continuous family of vector spaces $t \mapsto E(t) \subset \mathbb{R}^{2N}$ such that $E(t) \subset \ker(M(t) - \mathbb{1})$ and the crossing form $\Gamma(M, t)$ induces a nondegenerate quadratic form on $\ker(M(t)-\mathbb{1})/E(t)$. Assume ω_0 has constant rank on $E(t)$. Then*

$$\mu_{RS}(M) = \frac{1}{2}\mathrm{sign}\,\Gamma(M, 0) + \sum_{t:\dim F(t)\cap\ker(M(t)-\mathbb{1})>0} \mathrm{sign}\,\Gamma(M, t) + \frac{1}{2}\mathrm{sign}\,\Gamma(M, 1).$$

Proof. Let us first assume that the rank of ω_0 is constant equal to 0 on $E(t)$, i.e. $E(t)$ is isotropic. Let us decompose $\mathbb{R}^{2N} = E(t) \oplus J_0E(t) \oplus F(t)$, where $F(t)$ is the symplectic orthogonal of $E(t) \oplus J_0E(t)$. Given $\varepsilon > 0$ we denote by $\beta_\varepsilon : [0, 1] \to [0, \varepsilon]$ a smoothing of the function

$$t \mapsto \begin{cases} t, & 0 \le t \le \varepsilon, \\ \varepsilon, & \varepsilon \le t \le 1 - \varepsilon, \\ 1 - t, & 1 - \varepsilon \le t \le 1. \end{cases}$$

We define an element $\Phi^0_\varepsilon(t) \in \mathrm{Sp}(2N)$ which has the following matrix form with respect to the splitting $E(t) \oplus J_0E(t) \oplus F(t)$:

$$\Phi^0_\varepsilon(t) = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ \beta_\varepsilon(t) & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}.$$

We define $\widetilde{M}_\varepsilon(t) := M(t)\Phi^0_\varepsilon(t)$, and we have $\mu_{RS}(\widetilde{M}) = \mu_{RS}(M)$ since these paths are homotopic with fixed endpoints. We claim that the following equality holds for all $t \in]0,1[$:

$$\ker(\widetilde{M}(t) - \mathbb{1}) = \ker(M(t) - \mathbb{1}) \cap (J_0E(t) \oplus F(t)). \tag{A.2}$$

That $\ker(M(t) - \mathbb{1}) \cap (J_0E(t) \oplus F(t)) \subset \ker(\widetilde{M}(t) - \mathbb{1})$ follows from the fact that $\Phi^0_\varepsilon(t)$ acts by the identity on $J_0E(t) \oplus F(t)$. Conversely, let $v = v_1 + v_2 \in \ker(\widetilde{M}(t) - \mathbb{1})$, with $v_1 \in E(t)$ and $v_2 \in J_0E(t) \oplus F(t)$. The identity $\widetilde{M}(t)v = v$ is equivalent to $M(t)(v_1 + \beta_\varepsilon(t)J_0v_1 + v_2) = v_1 + v_2$, hence to $(M(t) - \mathbb{1})v_2 = -\beta_\varepsilon(t)M(t)J_0v_1$. Using that $M(t)v_1 = v_1$ we obtain

$$0 = \omega_0(v_1, (M(t) - \mathbb{1})v_2) = -\beta_\varepsilon(t)\omega_0(v_1, M(t)J_0v_1) = -\beta_\varepsilon(t)\omega_0(v_1, J_0v_1).$$

Since $\beta_\varepsilon(t) \neq 0$, this implies $v_1 = 0$, so that $v = v_2 \in J_0E(t) \oplus F(t)$ and $(M(t) - \mathbb{1})v_2 = (\widetilde{M}(t) - \mathbb{1})v_2 = 0$, as desired.

Since the restrictions of $M(t)$ and $\widetilde{M}(t)$ to $J_0E(t) \oplus F(t)$ are the same, it follows that the crossing form $\Gamma(\widetilde{M}, t)$ coincides with $\Gamma(M, t)$ on $\ker(\widetilde{M}(t) - \mathbb{1})$ for $t \in]0,1[$. On the other hand, a straightforward computation shows that

$$\begin{aligned} \operatorname{sign}\Gamma(\widetilde{M}, 0) &= \operatorname{sign}\Gamma(M, 0) + \dim E(0), \\ \operatorname{sign}\Gamma(\widetilde{M}, 1) &= \operatorname{sign}\Gamma(M, 1) - \dim E(1). \end{aligned} \tag{A.3}$$

Thus, the contributions at the endpoints compensate each other, and the conclusion follows using the definition of the Robbin-Salamon index via crossing forms.

We now assume that the rank of ω_0 on $E(t)$ is equal to $\dim E(t)$, i.e. $E(t)$ symplectic. Let us decompose $\mathbb{R}^{2N} = E(t) \oplus F(t)$, where $F(t)$ is the symplectic orthogonal of $E(t)$. Let $J(t)$ be a continuous family of complex structures on $E(t)$ which are compatible with ω_0. For $\varepsilon > 0$ we define a path $\Phi^1_\varepsilon : [0,1] \to \mathrm{Sp}(2N)$ whose matrix with respect to the decomposition $E(t) \oplus F(t)$ is

$$\Phi^1_\varepsilon(t) := \begin{pmatrix} \exp(J(t)\beta_\varepsilon(t)) & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

We denote $\widetilde{M}(t) := M(t)\Phi^1_\varepsilon(t)$, so that we have $\mu_{RS}(\widetilde{M}) = \mu_{RS}(M)$. We claim that

$$\ker(\widetilde{M}(t) - \mathbb{1}) = \ker(M(t) - \mathbb{1}) \cap F(t) \tag{A.4}$$

for all $t \in]0,1[$, whenever $0 < \varepsilon < \pi$. That $\ker(M(t)-\mathbb{1}) \cap F(t) \subset \ker(\widetilde{M}(t)-\mathbb{1})$ follows from the fact that $\Phi^1_\varepsilon(t)$ acts as the identity on $F(t)$. Conversely, let $v = v_1 + v_2 \in \ker(\widetilde{M}(t) - \mathbb{1})$ such that $v_1 \in E(t)$ and $v_2 \in F(t)$. The relation $\widetilde{M}(t)v = v$ is equivalent to $(M(t) - \mathbb{1})v_2 = (\mathbb{1} - \exp(J(t)\beta_\varepsilon(t)))v_1$. Then

$$\begin{aligned} 0 &= \omega_0(v_1, (M(t) - \mathbb{1})v_2) \\ &= \omega_0(v_1, (\mathbb{1} - \exp(J(t)\beta_\varepsilon(t)))v_1) \\ &= -\sin(\beta_\varepsilon(t))\omega_0(v_1, J(t)v_1). \end{aligned}$$

Since $\sin(\beta_\varepsilon(t)) \neq 0$, we obtain $v_1 = 0$ and the claim follows.

Since the restrictions of $M(t)$ and $\widetilde{M}(t)$ to $F(t)$ are the same, it follows that the crossing form $\Gamma(\widetilde{M}, t)$ coincides with $\Gamma(M, t)$ on $\ker(\widetilde{M}(t) - \mathbb{1})$ for $t \in]0,1[$. On the other hand, a straightforward computation shows that equations (A.3) still hold, and the conclusion follows.

Finally, we assume that the rank of ω_0 on $E(t)$ is lies strictly between 0 and dim $E(t)$. We choose a continuous splitting $E(t) = E_1(t) \oplus E_0(t)$ with $E_0(t) := E(t) \cap E(t)^{\omega_0}$ isotropic and $E_1(t) = E_0(t)^\perp$ symplectic. Here $E(t)^{\omega_0}$ denotes the symplectic orthogonal of $E(t)$, and $E_0(t)^\perp$ denotes the Euclidean orthogonal of $E_0(t)$ in $E(t)$. We decompose $\mathbb{R}^{2N} = E_1(t) \oplus E_0(t) \oplus J_0E_0(t) \oplus F(t)$, such that $F(t)$ is the symplectic orthogonal of $E_1(t) \oplus E_0(t) \oplus J_0E_0(t)$. Given $0 < \varepsilon < \pi$ we define as above two paths $\Phi^0_\varepsilon(t)$ acting as the identity on $E_1(t) \oplus F(t)$, and $\Phi^1_\varepsilon(t)$ acting as the identity on $E_0(t) \oplus J_0E_0(t) \oplus F(t)$. We denote $\widetilde{M}(t) := M(t)\Phi^0_\varepsilon(t)\Phi^1_\varepsilon(t)$, so that $\mu_{RS}(\widetilde{M}) = \mu_{RS}(M)$. One proves as above that the crossings of $\widetilde{M}$ and M on $]0,1[$ are the same, with the same crossing forms on $\ker(\widetilde{M}(t) - \mathbb{1})$, and moreover equations (A.3) still hold. This finishes the proof. □

Remark A.4. The crossing form $\Gamma(M, t)$ vanishes identically on $E(t)$. Indeed, given a path $v(t) \in E(t)$ we have $M(t)v(t) = v(t)$ and $\dot{M}(t)v(t) + M(t)\dot{v}(t) = \dot{v}(t)$. Dropping the t-variable for clarity, we have

$$\begin{aligned} \Gamma(M,t)(v(t)) &= \langle v, -J_0\dot{M}M^{-1}v\rangle \\ &= \langle v, -J_0(\dot{v} - M\dot{v})\rangle \\ &= -\langle v, J_0\dot{v}\rangle + \langle v, (M^{-1})^T J_0\dot{v}\rangle \\ &= -\langle v, J_0\dot{v}\rangle + \langle M^{-1}v, J_0\dot{v}\rangle \;=\; 0. \end{aligned}$$

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